2.3

Consolidate and Debrief

Key Concepts

- The number of permutations of n items is n factorial.

  $n! = n(n-1)(n-2)(n-3) \ldots \times 3 \times 2 \times 1$
- You can use factorials as a counting technique when repetition is not permitted.
- The number of r-permutations of n items can be calculated by

$$ {}_nP_r = n(n-1)(n-2)\cdots(n-r+1) $$
$$ = \frac{n!}{(n-r)!} $$

Reflect

R1. Which would have more possibilities, arranging r people from a group of n people **with** regard to order or **without** regard to order? Explain your reasoning.

R2. Use your calculator to determine the value of 0!. Explain why it would have this value. Include an example to support your explanation.

Practise

Choose the best answer for #4 and #5.

1. Evaluate.

a) 9!

b) $\frac{12!}{5!}$

c) ${}_7P_7$

d) ${}_8P_5$

2. Write in factorial form.

a) ${}_6P_4$

b) ${}_{15}P_6$

c) $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$

d) $8 \times 7 \times 6 \times 5$

e) $n(n-1)(n-2)(n-3)$

f) $(n+1) \times (n) \times (n-1) \times \ldots \times 3 \times 2 \times 1$

3. Express in the form ${}_nP_r$.

a) 6!

b) $91 \times 90 \times 89 \times 88 \times 87 \times 86$

c) $\frac{18!}{12!}$

4. Which is the correct simplification of $\frac{96!}{24!}$?

A 4!

B 4

C ${}_{96}P_{72}$

D ${}_{96}P_{24}$

5. Which is the correct number of permutations of five items from a list of nine items?

A 126

B 15 120

C 45

D 59 049

6. There are 15 teams competing in a synchronized swimming competition. In how many ways could first, second, and third place be awarded?

7. A club has 18 members. In how many ways could a president, vice president, treasurer, and secretary be elected?

Apply

8. There are 22 players on a baseball team. In how many ways could the batting order of nine players be assigned?

9. Write in simplest factorial form.
   a) $10 \times 9 \times 8 \times 7!$
   b) $99 \times 98 \times 97!$
   c) $90 \times 8!$
   d) $n(n-1)!$
   e) $(n+2)(n+1)n!$

10. **Application** A salesperson needs to visit 15 different offices during the week.
    a) In how many ways could this be done?
    b) In how many ways could she visit four different offices on Monday?
    c) In how many ways could she visit three different offices each day from Monday to Friday?

11. a) How many 10-digit numbers are there with no digits repeated?
    b) How many 7-digit numbers are there with no digits repeated?

12. Caleb needs to create an 8-digit password using only numbers. How many different passwords are there if he wants to use 00 exactly once?

✓ Achievement Check

13. The six members of the student council executive are lined up for a yearbook photo.
    a) In how many ways could the executive line up?
    b) In how many ways could this be done if the president and vice president must sit together?
    c) In how many ways could this be done if the president and vice president must sit together in the middle of the group?

14. How many ways are there to seat six boys and seven girls in a row of chairs so that none of the girls sit together?

15. **Thinking** Twenty figure skaters are in a competition. In the final round, the bottom five competitors skate first in a random order. The next five do likewise, and so on until the top five skate last in a random order. In how many ways could the skating order be assigned?

Extend

16. Solve for n.
    a) ${}_nP_2 = 110$
    b) $P(n, 3) = 5!$

17. Ten couples are being seated in a circle. How many different seating arrangements are there if each couple must sit together?

18. The names of the Knights of the Round Table at Winchester, UK, were engraved on the table, but they are no longer visible. There are 23 knights, plus King Arthur himself. In how many ways could King Arthur and the knights be seated at the Round Table?

19. A double factorial represents the product of all odd, or even, integers up to a given odd number, n. For example, $9!! = 1 \times 3 \times 5 \times 7 \times 9$.
    a) Express $9!!$ as a quotient of factorials.
    b) Express $(2k+1)!!$ as a quotient of factorials.
    c) Simplify $(2n)!!$, writing it in simple factorial form.

20. Without using a calculator, determine how many zeros occur at the end of $30!$.

2.4

Consolidate and Debrief

Key Concepts

- The rule of sum states that if one mutually exclusive event can occur in m ways, and a second can occur in n ways, then one **or** the other can occur in $m + n$ ways.
- If two events are not mutually exclusive, the principle of inclusion and exclusion needs to be considered:

  $n(A \text{ or } B) = n(A) + n(B) - n(A \text{ and } B)$.
- To reduce calculations, consider using the indirect method, which involves subtracting the unwanted event from the total number of outcomes in the sample space: $n(A) = n(S) - n(A')$.

Reflect

R1. A student council has five executive positions: president, vice president, secretary, treasurer, and assistant treasurer. There are six female and seven male candidates. It is important that at least one male and one female be on the executive. Explain why the indirect method is useful in determining the number of possible outcomes.

R2. Write a general guideline explaining when to use the fundamental counting principle and when to use the rule of sum for permutations. Include a similar example for each.

Practise

Choose the best answer for #3 and #4.

1. Determine the total number of arrangements of three or four toys from a basket of eight different toys.

2. a) How many ways are there to roll a sum of 7 or 11 on two dice?

b) How many ways are there to roll doubles or a sum divisible by three on two dice?

3. A game has players roll either one or two standard dice. Which is the total number of possible different outcomes?

A 42

B 36

C 18

D 12

4. Which is the total number of arrangements of the digits 1, 2, 3, 4, 5, if the even digits must not be together?

A 120

B 24

C 48

D 72

Apply

5. a) How many even numbers can be formed from the digits 1, 2, 3, 4, 5?

b) How many of these numbers are greater than 3000?

6. Application A motorcycle licence plate consists of two or three letters followed by four digits. How many licence plates can be made?

7. A security code consists of either five or six different letters. How many distinct security codes are possible?

8. **Communication** Suppose a country has a rule that a newborn child may have either one, two, or three names.

   a) If parents were to choose from a list of 50 names, how many choices would they have when naming their child?

   b) What if they could choose from 100 names?

   c) Explain why the total in part b) is more than 2 times the total in part a).

9. **Open Question** Five speakers, P, Q, R, S, and T, have been booked to address a meeting.

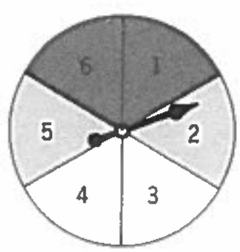
   a) In how many ways could the speakers be ordered if speaker P must go before speaker Q?

   b) Make up your own problem about these five speakers. Solve it, share it with a classmate, and check his or her solution.

10. How many five-digit numbers include the digits 4 or 6 or both?

11. Ten names are placed into a hat. In how many ways could they be pulled from the hat so they are not in alphabetical order?

12. A spinner has six equally spaced sections numbered 1 to 6 as shown. You spin the spinner four times.

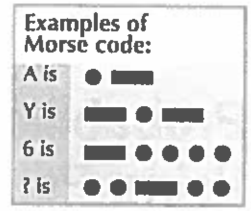


   a) In how many ways could the spinner result in the same colour on all four spins?

   b) In how many ways could the spinner result in an even number or the same colour on all four spins?

13. **Thinking** In the game of Monopoly, you can get out of jail by rolling doubles. If you are unsuccessful on the first roll you may try again, up to a total of three attempts. In how many ways could this occur? Explain your solution.

14. **Communication** Morse code uses dots and dashes to represent letters, digits, and eight punctuation symbols. Use the fundamental counting principle and the rule of sum to help explain why a maximum of six characters is needed.



✓ Achievement Check

15. A password must be 6, 7, or 8 characters long, and may include capital letters, lower-case letters, or digits. In how many ways could this be done

   a) with no restriction?

   b) with no repetition permitted?

   c) if at least one of the characters in part a) must be a digit?

Extend

16. How many different numbers can be formed by multiplying some or all of the numbers 2, 3, 4, 5, 6, 7, 8?

17. A derangement is a permutation of a set of numbers in which no item remains in its original position. For the set {1, 2, 3}, the derangements are {2, 3, 1} and {3, 1, 2}. The permutation {1, 3, 2} is not a derangement because 1 is in its original position. Determine the number of derangements of each set.

   a) {1, 2, 3, 4}

   b) {1, 2, 3, 4, 5}

18. The labels from six different cans of soup have come off. If you were to replace them at random, in how many ways could this be done so that

   a) none of the cans will be labelled correctly?

   b) at least one of the cans will be labelled correctly?

   c) all of the cans will be labelled correctly?

12. a) 45 configurations
    b) Adding a siding colour results in an increase of 15 choices. An additional trim colour results in only 9 more choices.
13. a) 2   b) 19   c) 4

Extend

14. There are 1 190 000 different local phone numbers Sarah can call.
15. Assume rows on checkerboard are numbered 0 to 7. The portion of the tree diagram that starts with a move diagonally left has 49 possible paths to the opposite side. Similarly, the portion of the tree diagram that starts with a move diagonally right has 54 possible paths to the opposite side. The total number of possible paths to the opposite side is 49 + 54, or 103.
16. a) 8   b) 16   c) 2^n

2.2 The Fundamental Counting Principle, pages 70–75

Example 1 Your Turn

60

Example 2 Your Turn

a) 308 915 776   b) 19 770 609 664

Example 3 Your Turn

13 800

Reflect

R1. Johnny is wrong. He should apply the fundamental counting principle: $4 \times 8 \times 3 = 96$.

R2. Answers may vary. The fundamental counting principle is the product of the number of ways multiple events can occur. For example, there are 3 flavours of ice cream and 6 choices of toppings to create a sundae. Event one, choose ice cream flavour, can happen in 3 ways. Event two, choosing a topping, can happen in 6 ways. The result in 3×6, or 18 different 1-topping ice cream sundaes.

Practise

1. a) 4   b) 8   c) 16   d) 2^n
2. a) 210   b) 2730
3. 240
4. a) 3   b) appetizers: 4, main course: 5, dessert: 3
   c) 60
5. C
6. B
7. a) 25   b) 20

Apply

8. a) 16   b) 64   c) 64   d) 4096
   e) 144   f) 248 832   g) n^k
9. 19 000
10. 90
11. 7776
12. a) 12 960 000   b) 11 703 240
13. a) 216 000   b) 205 320
15. a) 456 976 000   b) 17 576 000   c) 1 000 000
16. An Alberta licence plate will have much fewer choices than an Ontario licence plate. There are 26 choices for a letter, while there are only 10 choices for a digit.
17. 24
18. The same. In event one and two, the colour of the dice does not affect the choices for a die, and rolling three dice once has the same results as rolling one die three times.
19. Answers may vary.
    a) My security code is 325. I pressed ENTER 145 times before I saw my code. The actual number of possible outcomes for a three-digit security is 1000.
    b) It might take 100 times longer to break a five-digit code versus a three-digit code. The actual number of possible outcomes for a five-digit security is 100 000. Using a graphing calculator to randomly generate a five-digit code could possibly take more than 10 000 presses of ENTER because of duplicates or your code may never be generated.
20. Answers may vary. To find the number of choices for each of the three toppings, factor 4080: $2 \times 2 \times 2 \times 2 \times 3 \times 5 \times 17$. Using all the factors, create three values. For example, there could be $(2 \times 2 \times 2)$ choices for sauce, $(2 \times 3 \times 5)$ choices for actual topping ingredient, and 17 choices for cheese. Another possibility is that this includes four different sizes (S, M, L, XL). Then, there are actually 1020 $(2 \times 2 \times 3 \times 5 \times 17)$ topping options.
21. a) 1 048 576   b) 9 756 625

Extend

22. 52 400 215 plates
23. a) 60   b) 52
24. 5265
25. 1 275 120

2.3 Permutations and Factorials, pages 76–81

Example 1 Your Turn

a) 24   b) 720
c) 7920   d) 144

Example 2 Your Turn

40 320

Example 3 Your Turn

59 280

Example 4 Your Turn

240

Reflect

R1. Arranging r people from a group of n people with regard to order will have more possibilities. For example, ABC can be arranged in 6 ways: ABC, ACB, BAC, BCA, CAB, CBA. Without regard for order these arrangements are the same.

R2. Answers may vary. Using a calculator, 0! has a value of 1. Look at the formula for permutations. Suppose three people are awarded first, second, and third prize. So, $r = n = 3$ and the formula becomes $\frac{3!}{0!}$. In order for this to make sense, choose 0! to be defined as 1.

Practise

1. a) 362 880   b) 3 991 680   c) 5040   d) 6720
2. a) $\frac{6!}{2!}$ b) $\frac{15!}{9!}$ c) 7!   d) $\frac{8!}{4!}$
   e) $\frac{n!}{(n-4)!}$ f) $(n+1)!$
3. a) ${}_{6}P_{6}$ b) ${}_{91}P_{6}$ c) ${}_{18}P_{6}$
4. C   5. B   6. 2730   7. 73 440

Apply

8. ${}_{22}P_{9}$, or 180 503 769 600
9. a) 10!   b) 99!   c) 10!
   d) $n!$ e) $(n+2)!$

10. a) 15!, or 1 307 674 368 000 b) 32 760
c) 1 307 674 368 000
11. a) 3 265 920 b) 544 320
12. 3 720 087 14. 3 628 800 15. 207 3660 000

Extend

16. a) $n = 11$ b) $n = 6$
17. 371 589 120
18. 23!, or 25 852 016 738 884 976 640 000
19. a) $\frac{9!}{8!!}$ b) $\frac{(2k+1)!}{2k!!}$ c) $2^n n!$
20. 7

2.4 The Rule of Sum, pages 82–87

Example 1 Your Turn

a) 408 240
b) 46 080

Example 2 Your Turn

a) $_{13}P_3$
b) $_4P_3$
c) 1739

Example 3 Your Turn

480

Reflect

R1. It is simpler to calculate the number of executives without any males or females than all the possibilities for at least one male and one female.
R2. Use the fundamental counting principle when the events are independent. For example, rolling a die twice. The outcome of the first event does not affect the second. Use the rule of sum when events are mutually exclusive. For example, rolling a 1 or a 2. Both events cannot happen at the same time.

Practise

1. 2016
2. a) 8 b) 16
3. A 4. D

Apply

5. a) 130 b) 78
6. 182 520 000 7. 173 659 200
8. a) 120 100 b) 980 200
c) When the answers in parts a) and b) are expanded into factorial form, all three expressions in part b) are at least 2 times as big as those in part a). So, the result is more than $2^3 = 8$ times the answer in part a).
9. a) 60
b) Answers may vary. Question: Five speakers, P, Q, R, S, and T, are available to address a meeting. The organizer must decide whether to have four or five speakers. How many options would the organizer have for the meeting? Answer: 5! + 4! = 144. There are 144 options.
10. 61 328 11. 3 628 799
12. a) 48 b) 126
13. Answers may vary. Case 1: doubles on 1st roll = 6. Case 2: doubles on 2nd roll = 180. Case 3: doubles on 3rd roll = 5400. There are 5586 ways to get doubles.
14. Morse code is used to represent 26 letters, 10 digits, and 8 punctuation symbols, or a total of 44 symbols. Since each character has two options (dot or dash), a maximum of six characters is needed: $2^6 = 64$.

Extend

16. 82
17. a) 9 b) 44
18. a) 265 b) 455 c) 1

2.5 Probability Problems Using Permutations, pages 88–95

Example 1 Your Turn

a) $P(\text{all same}) = \frac{1}{10\,000\,000\,000}$
b) $P(\text{all 6s}) = \frac{1}{7776}$
For independent trials,
P(all the same) = $(P(\text{a success}))^{n \text{ trials}}$.

Example 2 Your Turn

$P(\text{in grade order}) = \frac{1}{24}$

Example 3 Your Turn

a) $P(\text{ace, ace, ace, jack, jack}) = \frac{1}{1\,082\,900}$
b) $P(\text{heart, heart, club, club, club}) = \frac{143}{166\,600}$

Example 4 Your Turn

a) approximately 0.7164 b) approximately 0.2836

Reflect

R1. No. The probability that at least two people have the same birthday is approximately 0.6269.
R2. Answers may vary. If the trials are dependent, permutations can be used. Look for restrictions such as, "without replacement" or "alphabetical order."
R3. Answers may vary. The first represents 3 of 12 objects being arranged. The second is 3 times 1 of 12 objects being arranged.

Practise

1. $P(\text{king, queen, jack}) = \frac{8}{16\,575}$
2. $\frac{1}{15}$ 3. A 4. C

Apply

5. $\frac{1\,307\,674\,367\,999}{1\,307\,674\,368\,000} : \frac{1}{1\,307\,674\,368\,000}$
6. a) approximately 0.000 505
b) $\frac{_{30}P_3}{_{365}P_3} \approx 0.000\,505$
7. a) $P(\text{doubles}) = \frac{1}{6}$ b) $P(\text{doubles twice}) = \frac{1}{36}$
c) They are the same.
8. a) $P(\text{3 boys}) = \frac{1}{8}$ b) $P(\text{4 boys}) = \frac{1}{16}$
c) $P(\text{5 boys}) = \frac{1}{32}$ d) $P(n \text{ boys}) = \frac{1}{2^n}$
9. a) $P(\text{MATH}) = \frac{1}{3024}$ b) $P(\text{M.A.T.H}) = \frac{1}{126}$
c) $P = \frac{4}{9}$
10. a) $P(\text{ascending order}) \approx 4.1697 \times 10^{-5}$
b) $P(\text{no same denomination}) \approx 0.2102$
11. $P(\text{at least two the same}) \approx 0.4114$
12. 23
13. a) $P(\text{songs in order}) = \frac{1}{3\,628\,800}$
b) $P = \frac{1}{45}$
14. a) 0.8203 : 0.1797 b) 0.4160 : 0.5840
15. Answers may vary. Example: 7