4.2 Integral Exponents

KEY IDEAS

- A power with a negative exponent can be written as a power with a positive exponent.

  - $a^{-n} = \frac{1}{a^n}, a \neq 0$ $\quad 2^{-5} = \frac{1}{2^5}$
  - $\frac{1}{a^{-n}} = a^n, a \neq 0$ $\quad \frac{1}{2^{-5}} = 2^5$

- You can apply the above principle to the exponent laws.

Exponent Law	Example
Note that a and b are rational or variable bases and m and n are integral exponents.	
Product of Powers $(a^m)(a^n) = a^{m+n}$	$(3^{-2})(3^4) = 3^{-2+4}$ $= 3^2$ or 9
Quotient of Powers $\frac{a^m}{a^n} = a^{m-n}, a \neq 0$	$\frac{x^3}{x^{-5}} = x^{3-(-5)}$ $= x^8$
Power of a Power $(a^m)^n = a^{mn}$	$(0.75^4)^{-2} = 0.75^{(4)(-2)}$ $= 0.75^{-8}$ or $\frac{1}{0.75^8}$
Power of a Product $(ab)^m = a^m b^m$	$(4z)^{-3} = \frac{1}{(4z)^3}$ or $\frac{1}{64z^3}$
Power of a Quotient $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}, b \neq 0$	$\left(\frac{t}{3}\right)^{-2} = \left(\frac{3}{t}\right)^2$ $= \frac{3^2}{t^2}$ or $\frac{9}{t^2}$
Zero Exponent $a^0 = 1, a \neq 0$	$(4y^2)^0 = 1$ $-(4y^2)^0 = -1$

Example

Write each expression as a power with a single, positive exponent. Then, evaluate where possible.

a) $\left(\frac{0.4^{-2}}{0.4^2}\right)$ b) $(6^4)(6^{-2})$ c) $[(3x)^{-2}]^{-3}$

Solution

a) **Method 1: Subtract the Exponents**

Since the bases are the same, you can subtract the exponents.

$$\left(\frac{0.4^{-2}}{0.4^2}\right) = 0.4^{(-2-2)}$$
$$= 0.4^{-4}$$
$$= 39.0625$$

Method 2: Use Positive Exponents

Convert the negative exponent to a positive exponent. Then, add the exponents when multiplying.

$$\left(\frac{0.4^{-2}}{0.4^2}\right) = \left(\frac{1}{0.4^2}\right)\left(\frac{1}{0.4^2}\right)$$
$$= \left(\frac{1}{0.4^{2+2}}\right)$$
$$= \left(\frac{1}{0.4^4}\right)$$
$$= 39.0625$$

b) Method 1: Add the Exponents
Since the bases are the same, you can add the exponents.

$$(6^4)(6^{-2}) = 6^{4+(-2)} = 6^2 = 36$$

Method 2: Use Positive Exponents
Convert the negative exponent to a positive exponent. Then, subtract the exponents when dividing.

$$(6^4)(6^{-2}) = (6^4)\left(\frac{1}{6^2}\right) = \frac{6^4}{6^2} = 6^{4-2} = 6^2 = 36$$

c) Method 1: Multiply the Exponents
Raise the power to the exponent. Then, multiply the exponents.

$$[(3x)^{-2}]^{-3} = (3x)^{(-2)(-3)} = (3x)^6 = 729x^6$$

Method 2: Use Positive Exponents
Convert the negative exponent to a positive exponent. Convert twice. Then, multiply the exponents.

$$[(3x)^{-2}]^{-3} = \left[\frac{1}{(3x)^2}\right]^{-3} = [(3x)^2]^3 = (3x)^{(2)(3)} = (3x)^6 = 729x^6$$

Hint: When an expression has a coefficient and a variable, apply the exponent law to each one. $(2b)^3 = (2^3)(b^3) = 8b^3$

A Practise

1. Write each expression with positive exponents.
   a) 4^{-2}
   b) $3x^{-3}$
   c) $(5x)^{-2}$
   d) $6a^{-3}b^{-2}$
   e) $-5a^{-4}$
   f) $-4a^4b^{-5}$
   g) $\left(\frac{2}{3}\right)^{-3}$
   h) $\frac{-3x^2}{y^{-4}}$
   i) $\frac{6a^{-3}}{b^4}$

2. Shelby rewrote the expression $\left(\frac{y^3}{4x^5}\right)^{-2}$ as $\frac{8x^{10}}{y^6}$. Is her answer correct? Justify your answer.

3. Simplify, then evaluate. Express your answers to four decimal places, if necessary.
   a) 1.4^{-3}
   b) $\left(\frac{-4^2}{2^3}\right)^{-3}$
   c) $[(2^{-2})(2^4)]^{-2}$
   d) $\left(\frac{-5^3}{5^3}\right)^{-3}$
   e) $\left(\frac{4}{4^3}\right)^{-3}$
   f) $\left(\frac{4^{-2}}{3^{-3}}\right)^2$

4. Simplify each expression by restating it using positive exponents only.
   a) a^4b^{-5}
   b) $\frac{-2}{a^3b^{-2}}$
   c) $[(p)^{-6}(p)^2]^{-3}$
   d) $\frac{12s^3}{4s^{-7}}$
   e) $(6x^{-4})^{-2}$
   f) $\left(\frac{t^{-3}}{t^5}\right)^{-2}$
   g) $[(n^3)(n^{-5})]^2$
   h) $(xy^{-3})^{-2}$

☆5. Simplify each expression. State the answer using positive exponents.

a) $(6)^{-3}(6)$

b) $\frac{(-2)^{-6}}{(-2)^{-3}}$

c) $\frac{3^3}{3^{-2}}$

d) $\left(\frac{4^0}{4^{-2}}\right)^2$

e) $(6^{-4})^2$

f) $-(3^4)^{-3}$

g) $[(2^4)(2^{-7})]^{-3}$

h) $\left(\frac{3^3}{4^3}\right)^{-2}$

i) $(4a^{-3})^{-2}$

j) $-3[(2^4)(2^{-3})]^{-2}$

6. The students in a grade 10 class were investigating the algae growth rate on the surface of a local lake. When they began, 425 cm^2 of the surface area of the lake was covered with algae. The amount of surface area covered with algae doubles each month. The students modelled this situation using the formula $SA = 425(2)^n$, where SA is the surface area of the lake covered in algae after n months. If conditions remain constant, how much of the lake will be covered in algae

a) after 6 months?

b) after 2 years?

7. A biologist is monitoring the population growth of caribou in a national park. There were 1400 caribou in 2010. The caribou population increases at a growth rate of 1.04% per year. The growth rate can be modelled using the formula $P = 1400(1.04)^n$, where P is the projected population after n years. Assuming that the growth rate remains constant, what would be the estimated caribou population in 2014?

B Apply

8. A culture of bacteria in a lab contains 400 bacterium cells. The number of cells doubles every hour. This situation can be modelled by the equation $B = 400(2)^h$, where B is the estimated number of bacteria and h is the time in hours. How many bacteria were present

a) after 3 h?

b) after 24 h?

c) 3 h ago?

☆9. Without using a calculator, evaluate $[((2^{-1})^2)^3]^{-1}$.

10. Kevin simplified $(2^3)(3^2)$ as 6^5. Is he correct? Justify your answer.

11. A radioactive element has a half-life of one month. The amount of the element remaining is given by the formula $A = 400\left(\frac{1}{2}\right)^n$, where n is the number of months. Today there are 400 g of the element.

a) How much will remain after 4 months?

b) How much was there a month ago?

☆12. The formula $d = \frac{1}{2}gt^2$ can be used to determine how long it takes an object to fall a certain distance from rest. In the formula, d is the distance the object falls, in metres, g is the acceleration due to gravity at 9.8 m/s^2, and t is the time it takes to fall, in seconds. Express each answer to one decimal place.

a) From what height does a penny fall if it takes 12.4 s to reach the ground?

b) How long does a penny take to fall from a height of 28.5 m?

c) How long does a penny take to reach the ground from a height of 248 m?

13. The population of Earth reached 6.8 billion people in 2009. Assume that the population increases by a growth rate of 1.8% per year and that the rate remains the same. The rate of growth can be modelled using the formula $P = [(6.8)(10^9)](1.018)^n$, where P is the estimated population and n is the number of years. Determine the projected population

a) by the end of 2015

b) by the end of 2020

14. In 2010, there were approximately 34 million people living in Canada. Assume that Canada's overall population growth rate is 0.9% per year and that the growth rate remains constant. The population can be estimated using the formula $P = [(3.4)(10^7)](1.009)^n$, where P is the estimated population and n is the number of years. What is the projected population

a) in 2018?

b) in 2021?

C Extend

☆15. Suppose you win the opportunity to receive a cash prize of $15 000 or double your money each year for a period of 25 years starting with an initial payment to you of $0.01. The value of your winnings can be determined using the formula $A = 0.01(2)^n$, where A is the payment at the end of n years.

a) What is the value of the payment you would receive after 3 years? after 10 years? after 25 years?

b) Which offer would you accept? Explain why.

c) If you received a cheque each year, how much money would you have received in total over the 25-year period?

16. The amount of sodium-24 remaining in a sample that started at 86 g can be represented by the equation $N = 86(0.5)^{\frac{t}{15}}$, where t is time, in hours. Determine the amount of sodium-24 remaining after each of the following time periods. Express the answers to two decimal places, if necessary.

a) after 30 h

b) after 90 h

c) after 120 h

17. Determine the value of x that makes each statement true.

a) $\left(\frac{4}{5}\right)^x = \frac{625}{256}$

b) $-3^x = -729$

c) $x^{-3} = \frac{27}{8}$

d) $2(6^x) = 432$

18. A scientist discovered a new isotope and called it mathodium-334. In the formula $A_f = A_i(3)^{-t}$, A_f represents the amount of the isotope remaining, A_i is the initial amount, in grams, and t is the time in days.

a) If a sample started at 85 g, how much would remain after 4 days? Express the answer to two decimal places.

b) The amount of mathodium-334 remaining after 6 h is 0.165 g. Calculate the amount of the original sample. Express the answer to two decimal places.

D Create Connections

19. Is $[(2^3)^4]^2$ equal to $[(2^4)^2]^3$? Justify your answer.

☆20. What value of x makes the following statement true?

$2^x + 2^x + 2^x + 2^x = 256$

21. Without using a calculator, show that $2^2 + 2^3 + 2^4$ is not equal to $(2^2)(2^3)(2^4)$. Explain why the answers are not the same.

22. Describe a real-life situation in which a positive exponent and a negative exponent can be used to model a problem.

a) Give an example of what the positive exponent represents.

b) Give an example of what the negative exponent represents.

23. a)

$\sqrt{25}$	5
$\sqrt{2.5}$	1.581...
$\sqrt{0.25}$	0.5
$\sqrt{0.025}$	0.158...
$\sqrt{0.0025}$	0.05
$\sqrt{0.00025}$	0.015...

b)

$\sqrt{81}$	9
$\sqrt{8.1}$	2.846...
$\sqrt{0.81}$	0.9
$\sqrt{0.081}$	0.284...
$\sqrt{0.0081}$	0.09
$\sqrt{0.00081}$	0.028...

c) Answers may vary. Look for the idea that a perfect decimal square exists if it has an even number of zeros before the perfect square number.

24. The expression $\sqrt{-25}$ is not a perfect square because when you multiply two positive or two negative numbers the answer is always positive. The expression $\sqrt[3]{-27}$ is a perfect cube because when you multiply three negative numbers, such as (−3)(−3)(−3), the answer is negative. Therefore, it is possible to have a negative perfect cube.

25. a) When you double the side lengths of a square, the area increases by a factor of 2^2 or 4. Example:

$A = s^2$
$= (2s)^2$
$= 4s^2$

When you triple the side lengths, the area increases by a factor of 3^2 or 9. Example:

$A = s^2$
$= (3s)^2$
$= 9s^2$

b) When you double the edge lengths of a cube, the volume increases by a factor of 2^3 or 8. Example:

$V = s^3$
$= (2s)^3$
$= 8s^3$

When you triple the edge lengths, the volume increases by a factor of 3^3 or 27. Example:

$V = s^3$
$= (3s)^3$
$= 27s^3$

4.2 Integral Exponents

1. a) $\frac{1}{4^2}$ b) $\frac{3}{x^3}$
c) $\frac{1}{(5x)^2}$ or $\frac{1}{25x^2}$ d) $\frac{6}{a^3b^2}$
e) $\frac{-5}{a^4}$ f) $\frac{-4a^4}{b^5}$
g) $\left(\frac{3}{2}\right)^3$ h) $-3x^2y^4$
i) $\frac{6}{a^3b^4}$

2. No. Shelby's answer is incorrect. The correct answer is $\frac{16x^{10}}{y^6}$.

3. a) 0.3644 b) −0.125
c) 0.0625 d) −1
e) 4096 f) 2.8477

4. a) $\frac{a^4}{b^5}$ b) $\frac{-2b^2}{a^3}$
c) p^{12} d) $3s^{10}$
e) $\frac{x^8}{6^2}$ f) t^{16}
g) $\frac{1}{n^4}$ h) $\frac{y^6}{x^2}$

5. a) $(6)^{-3}(6) = 6^{-3+1}$
$= 6^{-2}$
$= \frac{1}{6^2}$

b) $\frac{(-2)^{-6}}{(-2)^{-3}} = (-2)^{-6-(-3)}$
$= (-2)^{-3}$
$= \frac{1}{(-2)^3}$

c) $\frac{3^3}{3^{-2}} = 3^{3-(-2)}$
$= 3^5$

d) $\left(\frac{4^0}{4^{-2}}\right)^2 = (4^{0-(-2)})^2$
$= (4^2)^2$
$= 4^4$

e) $(6^{-4})^2 = 6^{(-4)(2)}$
$= 6^{-8}$
$= \frac{1}{6^8}$

f) $-(3^4)^{-3} = -(3)^{(4)(-3)}$
$= -(3)^{-12}$
$= \frac{-1}{(3)^{12}}$

g) $[(2^4)(2^{-7})]^{-3} = [(2)^{4+(-7)}]^{-3}$
$= [(2)^{-3}]^{-3}$
$= 2^{(-3)(-3)}$
$= 2^9$

h) $\left(\frac{3^3}{4^3}\right)^{-2} = \frac{(3)^{(3)(-2)}}{(4)^{(3)(-2)}}$
$= \frac{3^{-6}}{4^{-6}}$
$= \frac{4^6}{3^6}$

i) $(4a^{-3})^{-2} = (4)^{-2}a^{(-3)(-2)}$
$= (4)^{-2}a^6$
$= \frac{a^6}{4^2}$

j) $-3[(2^4)(2^{-3})]^{-2} = -3[(2)^{4+(-3)}]^{-2}$
$= -3[(2)^1]^{-2}$
$= -3(2)^{-2}$
$= \frac{-3}{2^2}$

6. a) 27 200 cm^2 b) 7 130 316 800 cm^2

7. approximately 1638 caribou

8. a) 3200 bacteria b) 6 710 886 400 bacteria
c) 50 bacteria

9. $\left[\left((2^{-1})^2\right)^3\right]^{-1} = \left[\left(\left(\frac{1}{2}\right)^2\right)^3\right]^{-1}$
$= \left[\left(\frac{1}{4}\right)^3\right]^{-1}$
$= \left[\frac{1}{64}\right]^{-1}$
$= 64$

Or, some students may evaluate as $2^6 = 64$.

10. No. Kevin is incorrect. Example: Since the bases are not the same, you cannot add the exponents. When simplified, the expression $(2^3)(3^2) = (8)(9) = 72$. The power $6^5 = 7776$.

11. a) 25 g b) 800 g

12. a) $d = \frac{1}{2}gt^2$
$= \frac{1}{2}(9.8)(12.4^2)$
$= (4.9)(153.76)$
$= 753.424$
The penny falls from a height of approximately 753.4 m.

b) $d = \frac{1}{2}gt^2$
$28.5 = \frac{1}{2}(9.8)t^2$
$28.5 = (4.9)t^2$
$\frac{28.5}{4.9} = t^2$
$t^2 = 5.816\,326\,5\ldots$
$t = \sqrt{5.816\,326\,5}$
$t = 2.411\,706\,1\ldots$
It takes approximately 2.4 s for the penny to fall.

c) $d = \frac{1}{2}gt^2$
$248 = \frac{1}{2}(9.8)t^2$
$248 = (4.9)t^2$
$\frac{248}{4.9} = t^2$
$t^2 = 50.612\,244\ldots$
$t = \sqrt{50.612\,244\ldots}$
$t = 7.114\,228\,2\ldots$
It takes approximately 7.1 s for the penny to fall.

13. a) approximately 7.6×10^9 or 7.6 billion people
b) approximately 8.3×10^9 or 8.3 billion people

14. a) approximately 3.65×10^7 or 36.5 million people
b) approximately 3.75×10^7 or 37.5 million people

15. a) $A = 0.01(2)^3 = 0.08$ After 3 years, the payment will be \$0.08.
$A = 0.01(2)^{10} = 10.24$ After 10 years, the payment will be \$10.24.
$A = 0.01(2)^{25} = 335\,544.32$. After 25 years, the payment will be \$335 544 32.

b) Accept any reasonable justification. Examples:
- I would accept the double the money offer because it is worth more over time.
- I would accept the cash prize because it is immediate and I have few financial resources at the present time.

c) Years 0–10 total = \$20.47; years 11–20 total = \$20 951.04; years 21–25 total = \$650 117.12. The total value over 25 years is \$671 088.63.

16. a) 21.5 g **b)** approximately 1.34 g
c) approximately 0.34 g

17. a) $x = -4$ **b)** $x = 6$
c) $x = \frac{2}{3}$ **d)** $x = 3$

18. a) approximately 1.05 g
b) approximately 0.22 g

19. Yes. Example: When you multiply the exponents within each expression, both are equal to 2^{24}.

20.
$$2^x + 2^x + 2^x + 2^x = 256$$
$$2^x(1 + 1 + 1 + 1) = 256$$
$$2^x = \frac{256}{4}$$
$$2^x = 64$$
$$2^x = 2^6$$
$$x = 6$$
or
$$2^x + 2^x + 2^x + 2^x = 256$$
$$2^x(4) = 256$$
$$2^x(2^2) = 256$$
$$2^{x+2} = 2^8$$
$$x + 2 = 8$$
$$x = 6$$

21. For $2^2 + 2^3 + 2^4$, use the order of operations to evaluate each power and then add the resulting values: 4 + 8 + 16 = 28. For $(2^2)(2^3)(2^4)$, since the powers have a common base, you can multiply by adding the exponents : $2^9 = 512$.

22. Example: calculating student enrollment at schools in the community.

a) You would use a positive exponent to predict enrollment in future years beyond the current year.

b) You would use a negative exponent to calculate student enrollment in years before the current year.

4.3 Rational Exponents

1. a) $a^{\frac{15}{2}}$ **b)** $y^{\frac{5}{6}}$
c) $x^{0.9}$ or $x^{\frac{9}{10}}$ **d)** $a^{0.6}$
e) x^{-4} or $\frac{1}{x^4}$ **f)** 9
g) $\frac{-4x^{\frac{1}{12}}}{3}$ **h)** $-10a^{\frac{11}{10}}$
i) $4a^{1.5}$ or $4a^{\frac{3}{2}}$

2. a) $\frac{1}{a^{\frac{5}{4}}}$ **b)** $\frac{1}{4}$
c) $y^{\frac{2}{3}}$ **d)** $\frac{1}{a^{\frac{5}{8}}}$
e) $a^{1.5}b^3$ or $a^{\frac{3}{2}}b^3$ **f)** $\frac{64x^{\frac{9}{4}}}{125}$
g) $\frac{3y^{\frac{2}{3}}}{2x^{\frac{3}{2}}}$ **h)** $\frac{3x^{\frac{1}{6}}}{5y^{\frac{3}{20}}}$

3. a)
$$\left(x^{\frac{2}{3}}\right)^q = x^{\frac{4}{3}}$$
$$x^{\frac{2q}{3}} = x^{\frac{4}{3}}$$
$$\frac{2q}{3} = \frac{4}{3}$$
$$2q = 4$$
$$q = 2$$
$$\left(x^{\frac{2}{3}}\right)^2 = x^{\frac{4}{3}}$$

b)
$$\left(x^{-\frac{2}{3}}\right)(x^q) = x^{-\frac{1}{6}}$$
$$x^{-\frac{2}{3} + q} = x^{-\frac{1}{6}}$$
$$\frac{-2}{3} + q = \frac{-1}{6}$$
$$q = \frac{-1}{6} + \frac{2}{3}$$
$$q = \frac{-1}{6} + \frac{4}{6}$$
$$q = \frac{3}{6} = \frac{1}{2}$$
$$\left(x^{-\frac{2}{3}}\right)\left(x^{\frac{1}{2}}\right) = x^{-\frac{1}{6}}$$