Section 2.1 Exercises

In Exercises 1–4, an object dropped from rest from the top of a tall building falls $y = 16t^2$ feet in the first t seconds.

1. Find the average speed during the first 3 seconds of fall.
2. Find the average speed during the first 4 seconds of fall.
3. Find the speed of the object at $t = 3$ seconds and confirm your answer algebraically.
4. Find the speed of the object at $t = 4$ seconds and confirm your answer algebraically.

In Exercises 5 and 6, use $\lim_{x \to c} k = k$, $\lim_{x \to c} x = c$, and the properties of limits to find the limit.

5. $\lim_{x \to c} (2x^3 - 3x^2 + x - 1)$

6. $\lim_{x \to c} \dfrac{x^4 - x^3 + 1}{x^2 + 9}$

In Exercises 7–14, determine the limit by substitution. Support graphically.

7. $\lim_{x \to -1/2} 3x^2(2x - 1)$ 8. $\lim_{x \to -4} (x + 3)^{1998}$

9. $\lim_{x \to 1} (x^3 + 3x^2 - 2x - 17)$ 10. $\lim_{y \to 2} \dfrac{y^2 + 5y + 6}{y + 2}$

11. $\lim_{y \to -3} \dfrac{y^2 + 4y + 3}{y^2 - 3}$ 12. $\lim_{x \to 1/2} \text{int } x$

13. $\lim_{x \to -2} (x - 6)^{2/3}$ 14. $\lim_{x \to 2} \sqrt{x + 3}$

In Exercises 15–18, explain why you cannot use substitution to determine the limit. Find the limit if it exists.

15. $\lim_{x \to -2} \sqrt{x - 2}$ 16. $\lim_{x \to 0} \dfrac{1}{x^2}$

17. $\lim_{x \to 0} \dfrac{|x|}{x}$ 18. $\lim_{x \to 0} \dfrac{(4 + x)^2 - 16}{x}$

In Exercises 19–28, determine the limit graphically. Confirm algebraically.

19. $\lim_{x \to 1} \dfrac{x - 1}{x^2 - 1}$ 20. $\lim_{t \to 2} \dfrac{t^2 - 3t + 2}{t^2 - 4}$

21. $\lim_{x \to 0} \dfrac{5x^3 + 8x^2}{3x^4 - 16x^2}$ 22. $\lim_{x \to 0} \dfrac{\frac{1}{2 + x} - \frac{1}{2}}{x}$

23. $\lim_{x \to 0} \dfrac{(2 + x)^3 - 8}{x}$ 24. $\lim_{x \to 0} \dfrac{\sin 2x}{x}$

25. $\lim_{x \to 0} \dfrac{\sin x}{2x^2 - x}$ 26. $\lim_{x \to 0} \dfrac{x + \sin x}{x}$

27. $\lim_{x \to 0} \dfrac{\sin^2 x}{x}$ 28. $\lim_{x \to 0} \dfrac{3 \sin 4x}{\sin 3x}$

In Exercises 29 and 30, use a graph to show that the limit does not exist.

29. $\lim_{x \to 1} \dfrac{x^2 - 4}{x - 1}$ 30. $\lim_{x \to 2} \dfrac{x + 1}{x^2 - 4}$

In Exercises 31–36, determine the limit.

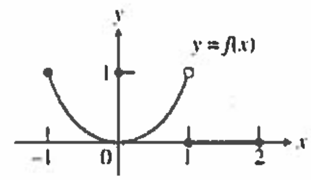
31. $\lim_{x \to 0^+} \text{int } x$ 32. $\lim_{x \to 0^-} \text{int } x$

33. $\lim_{x \to 0.01} \text{int } x$ 34. $\lim_{x \to 2^-} \text{int } x$

35. $\lim_{x \to 0^+} \dfrac{x}{|x|}$ 36. $\lim_{x \to 0^-} \dfrac{x}{|x|}$

In Exercises 37 and 38, which of the statements are true about the function $y = f(x)$ graphed there, and which are false?

37.



(a) $\lim_{x \to -1^+} f(x) = 1$ (b) $\lim_{x \to 0^-} f(x) = 0$

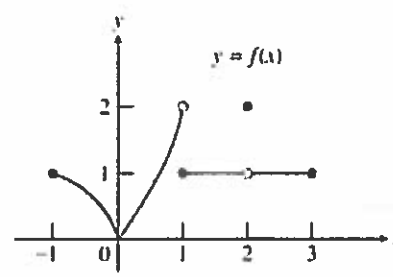
(c) $\lim_{x \to 0^-} f(x) = 1$ (d) $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$

(e) $\lim_{x \to 0} f(x)$ exists (f) $\lim_{x \to 0} f(x) = 0$

(g) $\lim_{x \to 0} f(x) = 1$ (h) $\lim_{x \to 1} f(x) = 1$

(i) $\lim_{x \to 1} f(x) = 0$ (j) $\lim_{x \to 2^-} f(x) = 2$

38.



(a) $\lim_{x \to -1^+} f(x) = 1$ (b) $\lim_{x \to 2} f(x)$ does not exist.

(c) $\lim_{x \to 2} f(x) = 2$ (d) $\lim_{x \to 1^-} f(x) = 2$

(e) $\lim_{x \to 1^+} f(x) = 1$ (f) $\lim_{x \to 1} f(x)$ does not exist.

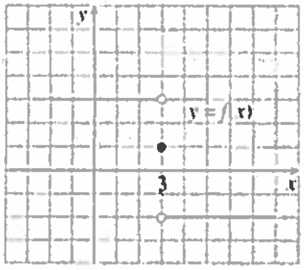
(g) $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x)$

(h) $\lim_{x \to c} f(x)$ exists at every c in $(-1, 1)$.

(i) $\lim_{x \to c} f(x)$ exists at every c in $(1, 3)$.

In Exercises 39–44, use the graph to estimate the limits and value of the function, or explain why the limits do not exist.

39.



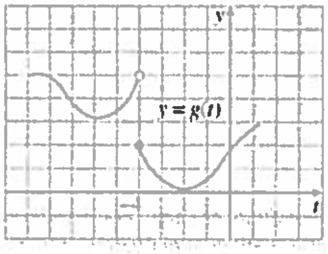
(a) $\lim_{x\to 3^-} f(x)$

(b) $\lim_{x\to 3^+} f(x)$

(c) $\lim_{x\to 3} f(x)$

(d) $f(3)$

40.



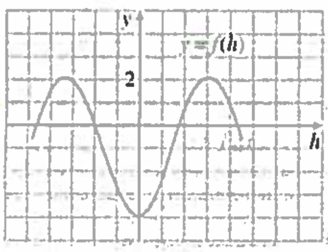
(a) $\lim_{t\to -4^-} g(t)$

(b) $\lim_{t\to -4^+} g(t)$

(c) $\lim_{t\to -4} g(t)$

(d) $g(-4)$

41.



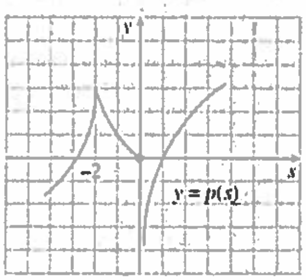
(a) $\lim_{h\to 0^-} f(h)$

(b) $\lim_{h\to 0^+} f(h)$

(c) $\lim_{h\to 0} f(h)$

(d) $f(0)$

42.



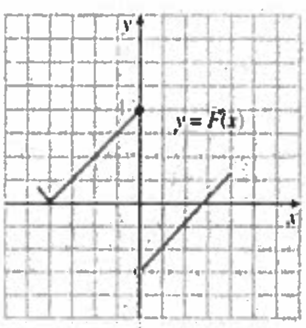
(a) $\lim_{s\to -2^-} p(s)$

(b) $\lim_{s\to -2^+} p(s)$

(c) $\lim_{s\to -2} p(s)$

(d) $p(-2)$

43.



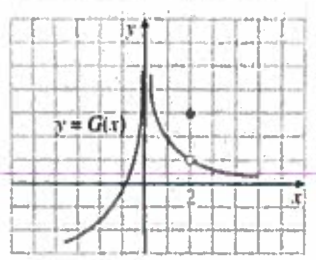
(a) $\lim_{x\to 0^-} F(x)$

(b) $\lim_{x\to 0^+} F(x)$

(c) $\lim_{x\to 0} F(x)$

(d) $F(0)$

44.



(a) $\lim_{x\to 2^-} G(x)$

(b) $\lim_{x\to 2^+} G(x)$

(c) $\lim_{x\to 2} G(x)$

(d) $G(2)$

In Exercises 45–48, match the function with the table.

45. $y_1 = \dfrac{x^2 + x - 2}{x - 1}$

46. $y_1 = \dfrac{x^2 - x - 2}{x - 1}$

47. $y_1 = \dfrac{x^2 - 2x + 1}{x - 1}$

48. $y_1 = \dfrac{x^2 + x - 2}{x + 1}$

(a)

X	Y₁
.7	-.4765
.8	-.3111
.9	-.1526
1	0
1.1	.14762
1.2	.29091
1.3	.43043
X = .7	

(b)

X	Y₁
.7	7.3667
.8	10.8
.9	20.9
1	ERROR
1.1	-18.9
1.2	-8.8
1.3	-5.367
X = .7	

(c)

X	Y₁
.7	2.7
.8	2.8
.9	2.9
1	ERROR
1.1	3.1
1.2	3.2
1.3	3.3
X = .7	

(d)

X	Y₁
.7	-.3
.8	-.2
.9	-.1
1	ERROR
1.1	.1
1.2	.2
1.3	.3
X = .7	

In Exercises 49 and 50, determine the limit.

49. Assume that $\lim_{x\to 4} f(x) = 0$ and $\lim_{x\to 4} g(x) = 3$.

(a) $\lim_{x\to 4} (g(x) + 3)$ (b) $\lim_{x\to 4} x f(x)$

(c) $\lim_{x\to 4} g^2(x)$ (d) $\lim_{x\to 4} \dfrac{g(x)}{f(x) - 1}$

50. Assume that $\lim_{x\to b} f(x) = 7$ and $\lim_{x\to b} g(x) = -3$.

(a) $\lim_{x\to b} (f(x) + g(x))$ (b) $\lim_{x\to b} (f(x) \cdot g(x))$

(c) $\lim_{x\to b} 4\, g(x)$ (d) $\lim_{x\to b} \dfrac{f(x)}{g(x)}$

In Exercises 51–54, complete parts (a), (b), and (c) for the piecewise-defined function.

(a) Draw the graph of f.

(b) Determine $\lim_{x\to c^+} f(x)$ and $\lim_{x\to c^-} f(x)$.

(c) **Writing to Learn** Does $\lim_{x\to c} f(x)$ exist? If so, what is it? If not, explain.

51. $c = 2,\ f(x) = \begin{cases} 3 - x, & x < 2 \\ \dfrac{x}{2} + 1, & x > 2 \end{cases}$

52. $c = 2,\ f(x) = \begin{cases} 3 - x, & x < 2 \\ 2, & x = 2 \\ x/2, & x > 2 \end{cases}$

53. $c = 1,\ f(x) = \begin{cases} \dfrac{1}{x - 1}, & x < 1 \\ x^3 - 2x + 5, & x \ge 1 \end{cases}$

54. $c = -1,\ f(x) = \begin{cases} 1 - x^2, & x \ne -1 \\ 2, & x = -1 \end{cases}$

In Exercises 55–58, complete parts (a)–(d) for the piecewise-defined function.

(a) Draw the graph of f.

(b) At what points c in the domain of f does $\lim_{x\to c} f(x)$ exist?

(c) At what points c does only the left-hand limit exist?

(d) At what points c does only the right-hand limit exist?

55. $f(x) = \begin{cases} \sin x, & -2\pi \le x < 0 \\ \cos x, & 0 \le x \le 2\pi \end{cases}$

56. $f(x) = \begin{cases} \cos x, & -\pi \le x < 0 \\ \sec x, & 0 \le x \le \pi \end{cases}$

57. $f(x) = \begin{cases} \sqrt{1 - x^2}, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \\ 2, & x = 2 \end{cases}$

58. $f(x) = \begin{cases} x, & -1 \le x < 0, \text{ or } 0 < x \le 1 \\ 1, & x = 0 \\ 0, & x < -1, \text{ or } x > 1 \end{cases}$

In Exercises 59–62, find the limit graphically. Use the Sandwich Theorem to confirm your answer.

59. $\lim_{x\to 0} x \sin x$

60. $\lim_{x\to 0} x^2 \sin x$

61. $\lim_{x\to 0} x^2 \sin \dfrac{1}{x^2}$

62. $\lim_{x\to 0} x^2 \cos \dfrac{1}{x^2}$

63. ***Free Fall*** A water balloon dropped from a window high above the ground falls $y = 4.9t^2$ m in t sec. Find the balloon's

(a) average speed during the first 3 sec of fall.

(b) speed at the instant $t = 3$.

64. ***Free Fall on a Small Airless Planet*** A rock released from rest to fall on a small airless planet falls $y = gt^2$ m in t sec, g a constant. Suppose that the rock falls to the bottom of a crevasse 20 m below and reaches the bottom in 4 sec.

(a) Find the value of g.

(b) Find the average speed for the fall.

(c) With what speed did the rock hit the bottom?

Standardized Test Questions

You should solve the following problems without using a graphing calculator.

65. True or False If $\lim_{x\to c^-} f(x) = 2$ and $\lim_{x\to c^+} f(x) = 2$, then $\lim_{x\to c} f(x) = 2$. Justify your answer.

66. True or False $\lim_{x\to 0} \dfrac{x + \sin x}{x} = 2$. Justify your answer.

In Exercises 67–70, use the following function.

$$f(x) = \begin{cases} 2 - x, & x \le 1 \\ \dfrac{x}{2} + 1, & x > 1 \end{cases}$$

67. Multiple Choice What is the value of $\lim_{x\to 1^-} f(x)$?

(A) 5/2 (B) 3/2 (C) 1 (D) 0 (E) does not exist

68. Multiple Choice What is the value of $\lim_{x\to 1^+} f(x)$?

(A) 5/2 (B) 3/2 (C) 1 (D) 0 (E) does not exist

69. Multiple Choice What is the value of $\lim_{x\to 1} f(x)$?

(A) 5/2 (B) 3/2 (C) 1 (D) 0 (E) does not exist

70. Multiple Choice What is the value of $f(1)$?

(A) 5/2 (B) 3/2 (C) 1 (D) 0 (E) does not exist

Explorations

In Exercises 71–74, complete the following tables and state what you believe $\lim_{x\to 0} f(x)$ to be.

(a)

x	-0.1	-0.01	-0.001	-0.0001	...
$f(x)$	?	?	?	?	

(b)

x	0.1	0.01	0.001	0.0001	...
$f(x)$	?	?	?	?	

71. $f(x) = x \sin \dfrac{1}{x}$

72. $f(x) = \sin \dfrac{1}{x}$

73. $f(x) = \dfrac{10^x - 1}{x}$

74. $f(x) = x \sin (\ln |x|)$

75. Group Activity To prove that $\lim_{\theta\to 0} (\sin \theta)/\theta = 1$ when θ is measured in radians, the plan is to show that the right- and left-hand limits are both 1.

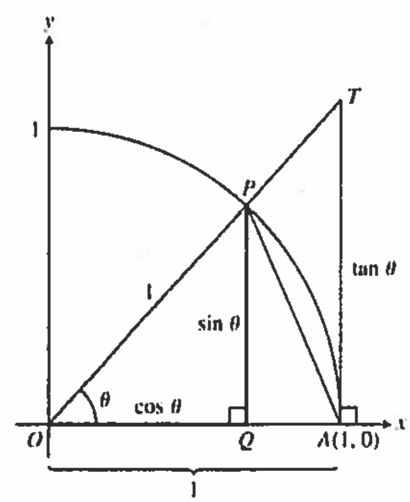
(a) To show that the right-hand limit is 1, explain why we can restrict our attention to $0 < \theta < \pi/2$.

(b) Use the figure to show that

$$\text{area of } \Delta OAP = \frac{1}{2} \sin \theta,$$

$$\text{area of sector } OAP = \frac{\theta}{2},$$

$$\text{area of } \Delta OAT = \frac{1}{2} \tan \theta.$$



(c) Use part (b) and the figure to show that for $0 < \theta < \pi/2$,

$$\frac{1}{2} \sin \theta < \frac{1}{2}\theta < \frac{1}{2} \tan \theta.$$

(d) Show that for $0 < \theta < \pi/2$ the inequality of part (c) can be written in the form

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$$

(e) Show that for $0 < \theta < \pi/2$ the inequality of part (d) can be written in the form

$$\cos\theta < \frac{\sin\theta}{\theta} < 1.$$

(f) Use the Sandwich Theorem to show that

$$\lim_{\theta\to 0^+} \frac{\sin\theta}{\theta} = 1.$$

(g) Show that $(\sin\theta)/\theta$ is an even function.

(h) Use part (g) to show that

$$\lim_{\theta\to 0^-} \frac{\sin\theta}{\theta} = 1.$$

(i) Finally, show that

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1.$$

Extending the Ideas

76. ***Controlling Outputs*** Let $f(x) = \sqrt{3x - 2}$.

(a) Show that $\lim_{x\to 2} f(x) = 2 = f(2)$.

(b) Use a graph to estimate values for a and b so that $1.8 < f(x) < 2.2$ provided $a < x < b$.

(c) Use a graph to estimate values for a and b so that $1.99 < f(x) < 2.01$ provided $a < x < b$.

77. ***Controlling Outputs*** Let $f(x) = \sin x$.

(a) Find $f(\pi/6)$.

(b) Use a graph to estimate an interval (a, b) about $x = \pi/6$ so that $0.3 < f(x) < 0.7$ provided $a < x < b$.

(c) Use a graph to estimate an interval (a, b) about $x = \pi/6$ so that $0.49 < f(x) < 0.51$ provided $a < x < b$.

78. ***Limits and Geometry*** Let $P(a, a^2)$ be a point on the parabola $y = x^2$, $a > 0$. Let O be the origin and $(0, b)$ the y-intercept of the perpendicular bisector of line segment OP. Find $\lim_{P\to O} b$.