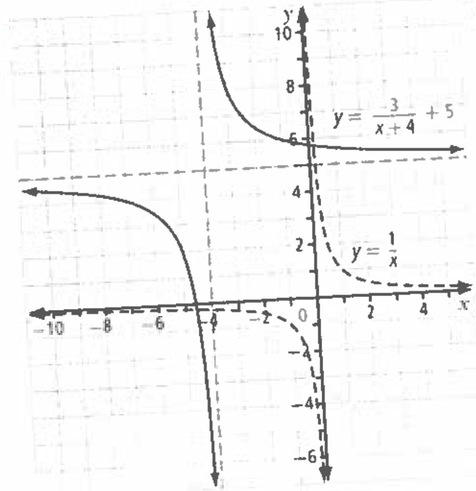
KEY

Chapter 9 Rational Functions

9.1 Exploring Rational Functions Using Transformations

KEY IDEAS

- Rational functions are functions of the form $y = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomial expressions and $q(x) \neq 0$.
- You can graph a rational function by creating a table of values and then graphing the points in the table. To create a table of values,
  - identify the non-permissible value(s)
  - write the non-permissible value in the middle row of the table
  - enter positive values above the non-permissible value and negative values below the non-permissible value
  - choose small and large values of x to give you a spread of values
- You can use what you know about the base function $y = \frac{1}{x}$ and transformations to graph equations of the form $y = \frac{a}{x - h} + k$.



Example:

For $y = \frac{3}{x + 4} + 5$, the values of the parameters are

$a = 3$, representing a vertical stretch by a factor of 3
$h = 4$, representing a horizontal translation 4 units to the left
$k = 5$, representing a vertical translation 5 units up
vertical asymptote: $x = -4$
horizontal asymptotes: $y = 5$

- Some equations of rational functions can be manipulated algebraically into the form $y = \frac{a}{x - h} + k$ by creating a common factor in the numerator and the denominator.

Example:

$$y = \frac{3x + 6}{x - 4}$$

$$y = \frac{3x - 12 + 12 + 6}{x - 4}$$

$$y = \frac{3x - 12 + 18}{x - 4}$$

$$y = \frac{3(x - 4)}{x - 4} + \frac{18}{x - 4}$$

$$y = \frac{18}{x - 4} + 3$$

Working Example 3: Graph a Rational Function With Linear Expressions in the Numerator and the Denominator

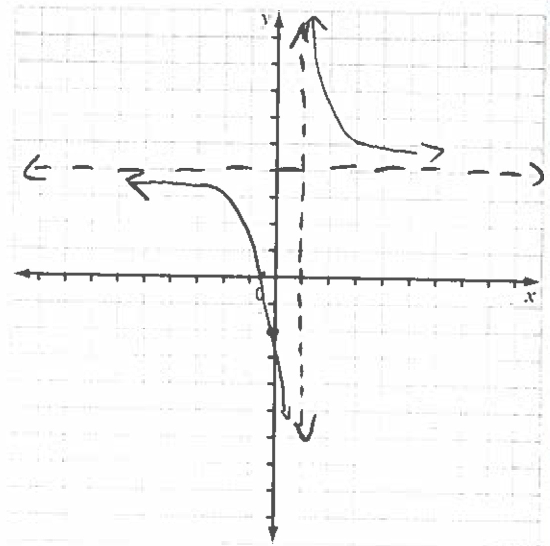
Graph $y = \frac{4x + 2}{x - 1}$. Identify any asymptotes and intercepts.

Solution

Let $x = 0$. Solve for y to determine the y-intercept.

$y = \frac{4(0)+2}{0-1} = \frac{2}{-1}$

The y-intercept is at (0, $\underline{-2}$).



Let $y = 0$. Solve for x to determine the x-intercept.

$$0 = \frac{4x + 2}{x - 1}$$

$$(\underline{x-1})(0) = (\underline{x-1})\frac{4x + 2}{x - 1}$$

$$\underline{0} = \underline{4x+2}$$

$$\underline{-2} = 4x$$

$$\underline{-\tfrac{1}{2}} = x$$

The x-intercept is at ($\underline{-0.5}$, 0).

∇ 0 Manipulate the equation of the function algebraically to obtain the form $y = \frac{a}{x - h} + k$.

$$y = \frac{4x + 2}{x - 1}$$

$$y = \frac{4x - 4 + 4 + 2}{x - 1}$$

$$y = \frac{4(x-1) + 6}{x-1}$$

$$y = 4(1) + \frac{6}{x-1} \rightarrow f(x) = \frac{6}{x-1} + 4$$

Why is 4 subtracted and added to the numerator?

Which parameters determine the vertical and horizontal asymptotes of the transformed function?

VA: $x = 1$ ("h")
HA: $y = 4$ ("k")

The parameters are $a = \underline{6}$, $h = \underline{1}$, and $k = \underline{4}$. State the effect of each parameter on the graph of $y = \frac{1}{x}$. Then, use the information you have generated to sketch the transformed function on the grid above.

→ HT right by 1 unit
↑ VT up by 4 units

$a = 6$: VS by a factor of 6

To see a similar example, see Example 3 on pages 435–437 of *Pre-Calculus 12*.

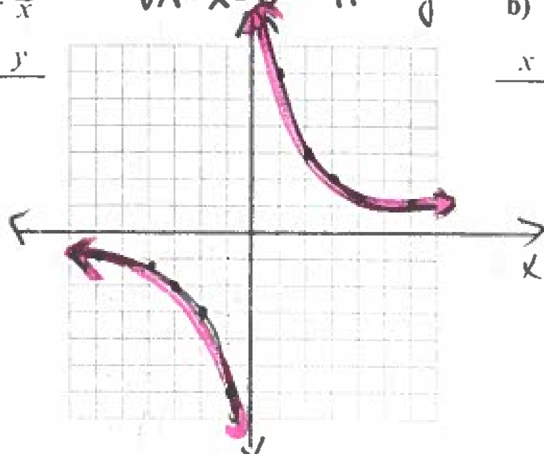
Check Your Understanding

Practise

1. Graph each function using a table of values. Identify the asymptotes.

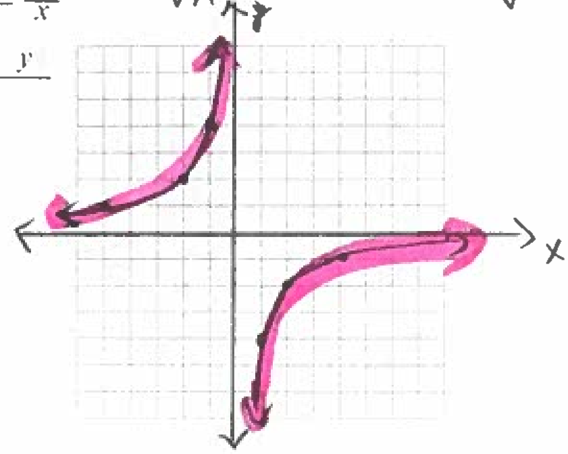
a) $y = \frac{6}{x}$ VA: $x=0$ HA: $y=0$

x	y



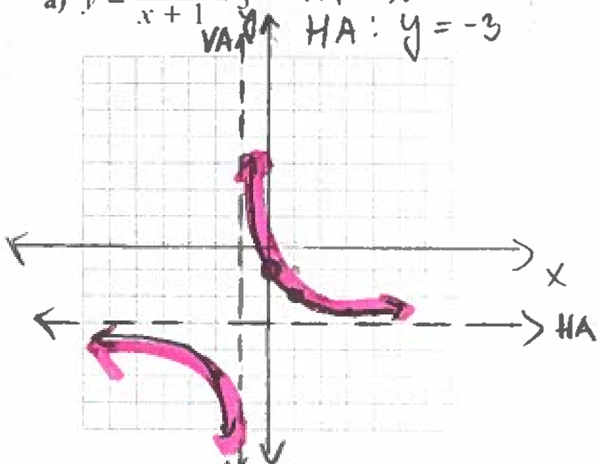
b) $y = \frac{-4}{x}$ VA: $x=0$ HA: $y=0$

x	y

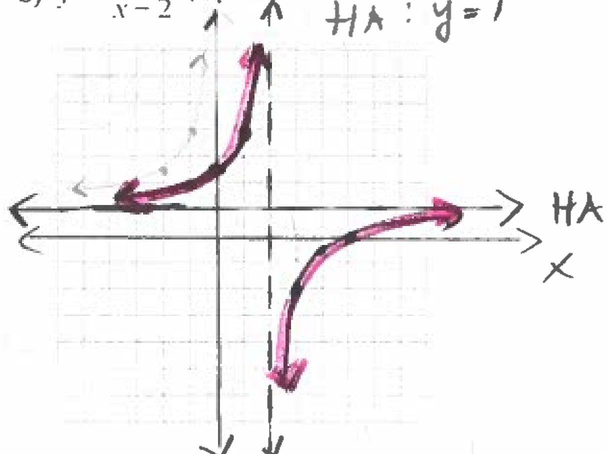


2. Graph each function using transformations. Label the asymptotes and intercepts.

a) $y = \frac{2}{x+1} - 3$ VA: $x=-1$ HA: $y=-3$



b) $y = \frac{-3}{x-2} + 1$ VA: $x=2$ HA: $y=1$

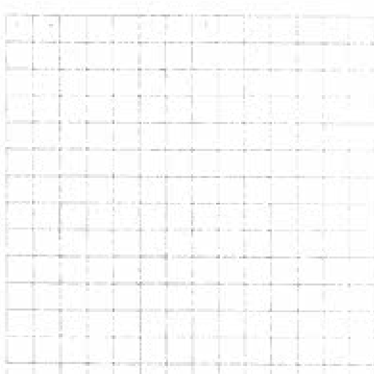


3. Graph each function using technology. Identify any asymptotes and intercepts. Sketch the graph.

a) $y = \frac{2x+1}{x-2} = \frac{2x-4+4+1}{x-2} = \frac{2(x-2)+5}{(x-2)}$

$= 2(1) + \frac{5}{x-2}$

$= \boxed{\frac{5}{x-2} + 2}$

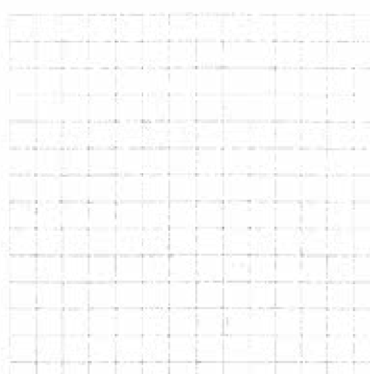


b) $y = \frac{-3x-4}{x+2} = \frac{-3x-2-4+2}{x+2} = \frac{-3x-6+2}{x+2}$

$= \frac{-3(x+2)+2}{(x+2)}$

$= -3 + \frac{2}{x+2}$

$= \boxed{\frac{2}{x+2} - 3}$



Apply

4. Match each graph with its equation.

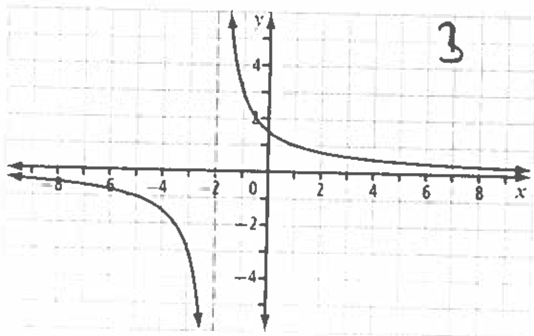
a) $y = \frac{3}{x-2}$ → HA: x = 2

b) $y = \frac{3}{x+2}$ VA: x = -2

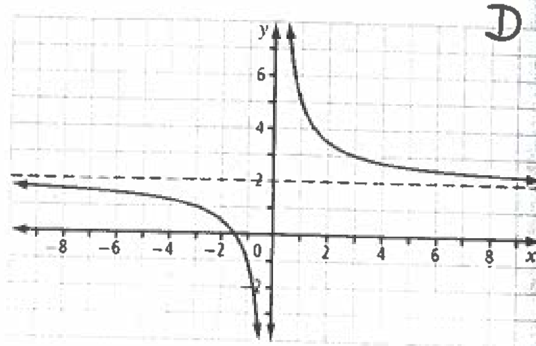
c) $y = \frac{3}{x} - 2$ → VA: y = -2

d) $y = \frac{3}{x} + 2$ VA: y = 2

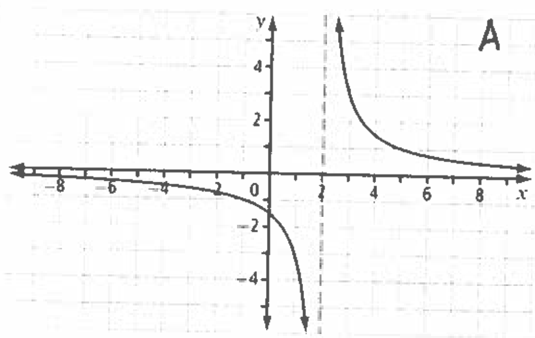
A



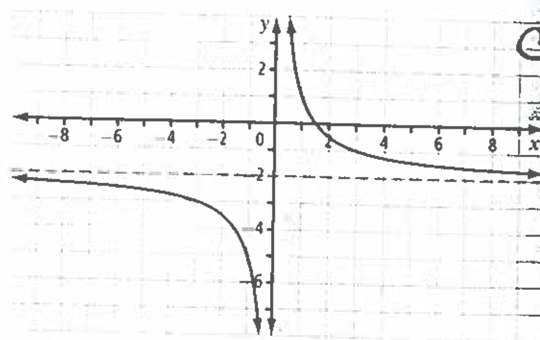
B



C

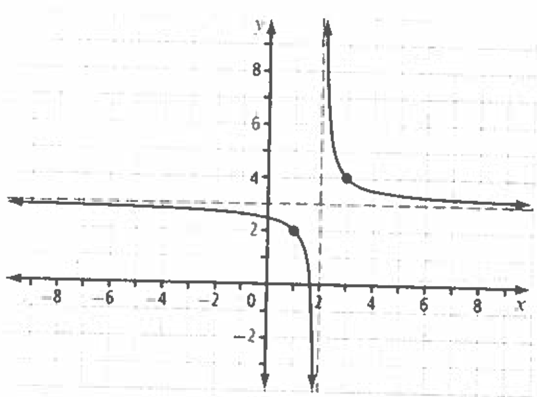


D



5. Write the equation of each function in the form of $y = \frac{a}{x-h} + k$.

a)



VA: x=2 no VS

HA: y = 3

For the graph of $y = \frac{1}{x}$, what is the relationship between the intersection of the asymptotes and the point (1, 1)? How can you use this knowledge to determine a vertical stretch?

The vertical asymptote provides the __h__ parameter. The horizontal asymptote provides the __k__ parameter.

$$\boxed{y = \frac{1}{x-2} + 3}$$

VSE by a function

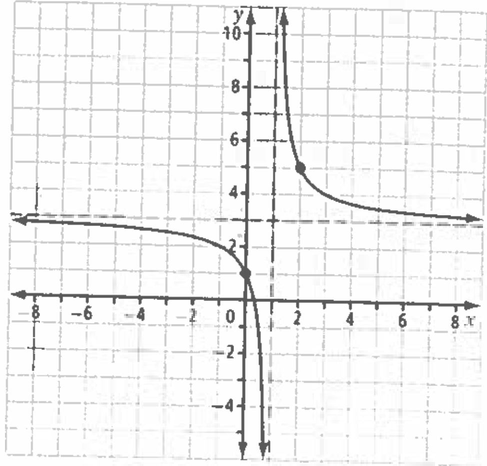
HA: $y = 3$ VA: $x = 1$

HA: $y = -4$

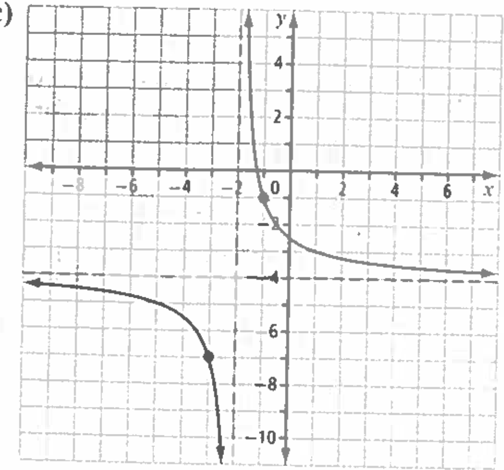
VA: $x = -2$

$\boxed{a = 3}$

b)



c)



$$y = \frac{2}{x-1} + 3$$

$$y = \frac{3}{x+2} - 4$$

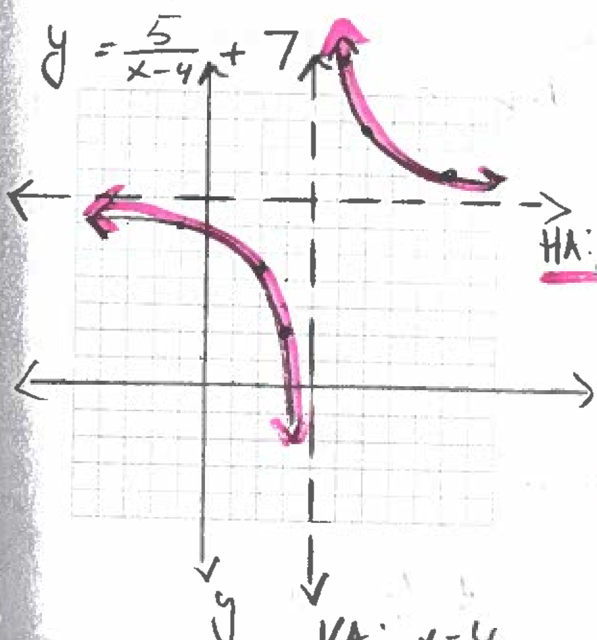
6. Write each equation in the form $y = \dfrac{a}{x-h} + k$. Then, graph the function using transformations. Indicate the asymptotes.

a) $y = \dfrac{7x - 23}{x - 4}$

$$y = \frac{7x - 23 - 5 + 5}{x - 4}$$

$$y = \frac{7x - 28 + 5}{x - 4} = \frac{7(x-4)+5}{x-4}$$

$$y = \frac{5}{x-4} + 7$$



HA: $y=7$

VA: $x=4$

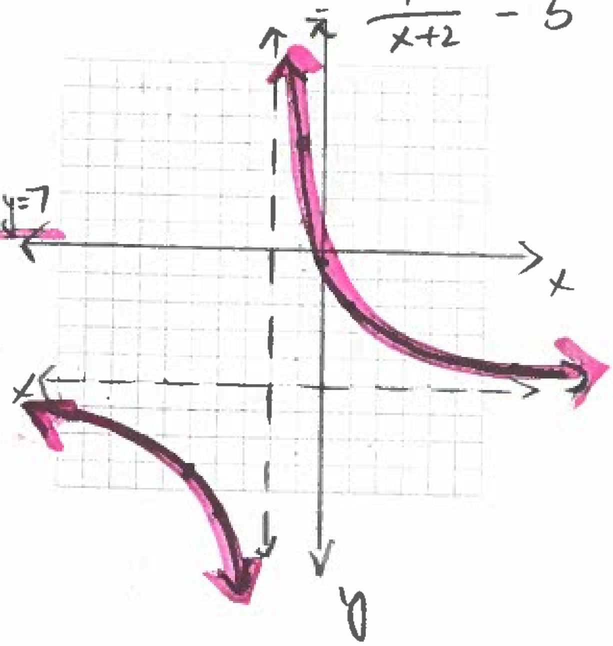
b) $y = \dfrac{-5x - 1}{x + 2}$

$$= \frac{-5x - 1 - 9 + 9}{x+2}$$

$$= \frac{-5x - 10 + 9}{x+2}$$

$$= \frac{-5(x+2) + 9}{x+2}$$

$$= \frac{9}{x+2} - 5$$



There are many possibilities; including functions reflected in the x-axis, and VSE/C

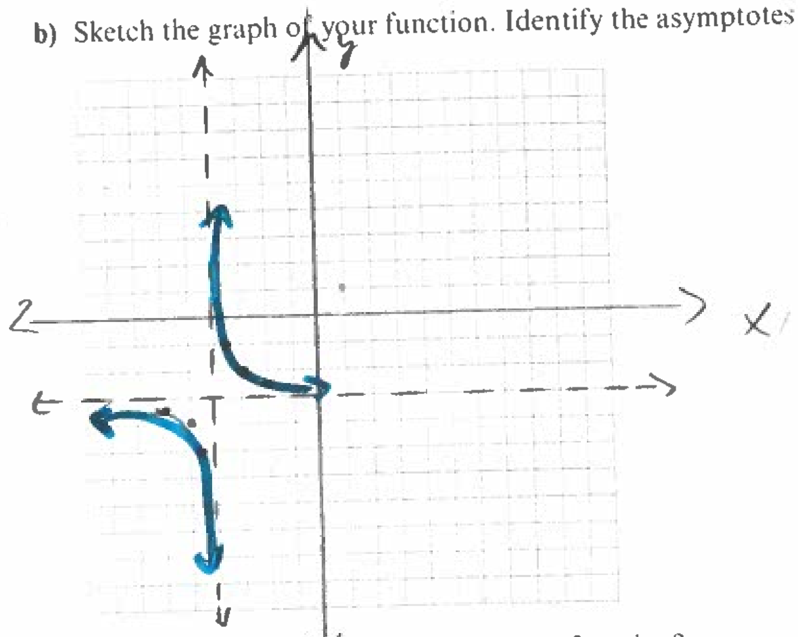
$$y = \frac{-1}{x+4} - 3$$

Connect

7. a) Determine an equation of a rational function that has an asymptote at $x = -3$ and $y = -4$. Explain the rationale for your equation.

$$y = \frac{1}{x+4} - 3$$

b) Sketch the graph of your function. Identify the asymptotes on your graph.



c) What is the domain and range of your function?

D: $\{x \mid x \neq -4, x \in \mathbb{R}\}$ and R: $\{y \mid y \neq -3, y \in \mathbb{R}\}$

d) Is there another possible function with these asymptotes? Explain.

Many possibilities as long as $h = -4$ and $k = -3$.

8. Describe the similarities and differences between graphing $y = \frac{2}{x-4} - 3$, $y = 2(x-4)^2 - 3$, and $y = 2\sqrt{x-4} - 3$ without technology.

→ applying same transformations: HT right by 4 units and VT down by 3 units.

$(x, y) \to (x+4, y-3)$ for all 3 functions

↑

ordered pairs are different