Chapter 1 Overview

This chapter reviews the most important things you need to know to start learning calculus. It also introduces the use of a graphing utility as a tool to investigate mathematical ideas, to support analytic work, and to solve problems with numerical and graphical methods. The emphasis is on functions and graphs, the main building blocks of calculus.

Functions and parametric equations are the major tools for describing the real world in mathematical terms, from temperature variations to planetary motions, from brain waves to business cycles, and from heartbeat patterns to population growth. Many functions have particular importance because of the behavior they describe. Trigonometric functions describe cyclic, repetitive activity; exponential, logarithmic, and logistic functions describe growth and decay; and polynomial functions can approximate these and most other functions.

1.1 Lines

What you'll learn about

- Increments
- Slope of a Line
- Parallel and Perpendicular Lines
- Equations of Lines
- Applications

... and why

Linear equations are used extensively in business and economic applications.

Increments

One reason calculus has proved to be so useful is that it is the right mathematics for relating the rate of change of a quantity to the graph of the quantity. Explaining that relationship is one goal of this book. It all begins with the slopes of lines.

When a particle in the plane moves from one point to another, the net changes or *increments* in its coordinates are found by subtracting the coordinates of its starting point from the coordinates of its stopping point.

DEFINITION Increments

If a particle moves from the point (x_1, y_1) to the point (x_2, y_2), the **increments** in its coordinates are

$$\Delta x = x_2 - x_1 \quad \text{and} \quad \Delta y = y_2 - y_1.$$

The symbols Δx and Δy are read "delta x" and "delta y." The letter Δ is a Greek capital d for "difference." Neither Δx nor Δy denotes multiplication; Δx is not "delta times x" nor is Δy "delta times y."

Increments can be positive, negative, or zero, as shown in Example 1.

EXAMPLE 1 Finding Increments

The coordinate increments from $(4, -3)$ to $(2, 5)$ are

$$\Delta x = 2 - 4 = -2, \quad \Delta y = 5 - (-3) = 8.$$

From $(5, 6)$ to $(5, 1)$, the increments are

$$\Delta x = 5 - 5 = 0, \quad \Delta y = 1 - 6 = -5. \quad \textit{Now try Exercise 1.}$$
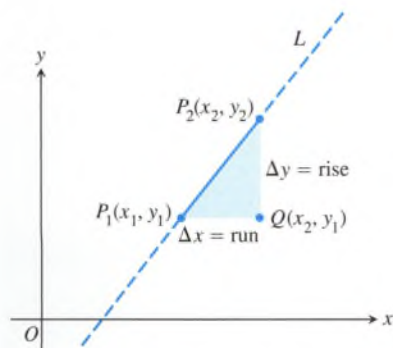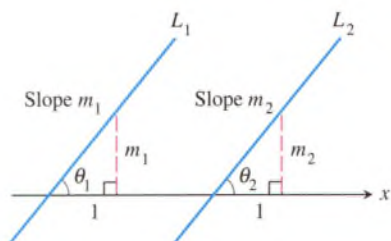

Figure 1.1 The slope of line L is
$$m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x}.$$
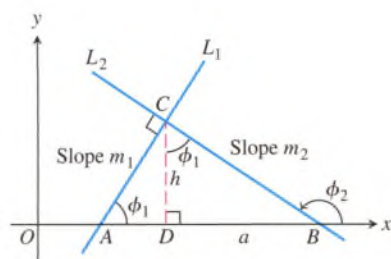
Slope of a Line

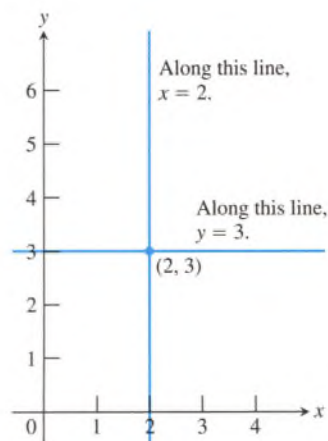Each nonvertical line has a *slope,* which we can calculate from increments in coordinates.

Let L be a nonvertical line in the plane and $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ two points on L (Figure 1.1). We call $\Delta y = y_2 - y_1$ the **rise** from P_1 to P_2 and $\Delta x = x_2 - x_1$ the **run** from

P_1 to P_2. Since L is not vertical, $\Delta x \neq 0$ and we define the slope of L to be the amount of rise per unit of run. It is conventional to denote the slope by the letter m.

Figure 1.2 If $L_1 \parallel L_2$, then $\theta_1 = \theta_2$ and $m_1 = m_2$. Conversely, if $m_1 = m_2$, then $\theta_1 = \theta_2$ and $L_1 \parallel L_2$.

> **DEFINITION Slope**
>
> Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be points on a nonvertical line, L. The **slope** of L is
> $$m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}.$$

A line that goes uphill as x increases has a positive slope. A line that goes downhill as x increases has a negative slope. A horizontal line has slope zero since all of its points have the same y-coordinate, making $\Delta y = 0$. For vertical lines, $\Delta x = 0$ and the ratio $\Delta y/\Delta x$ is undefined. We express this by saying that vertical lines *have no slope*.

Parallel and Perpendicular Lines

Parallel lines form equal angles with the x-axis (Figure 1.2). Hence, nonvertical parallel lines have the same slope. Conversely, lines with equal slopes form equal angles with the x-axis and are therefore parallel.

If two nonvertical lines L_1 and L_2 are perpendicular, their slopes m_1 and m_2 satisfy $m_1 m_2 = -1$, so each slope is the *negative reciprocal* of the other:

$$m_1 = -\frac{1}{m_2}, \qquad m_2 = -\frac{1}{m_1}.$$

Figure 1.3 $\triangle ADC$ is similar to $\triangle CDB$. Hence ϕ_1 is also the upper angle in $\triangle CDB$, where $\tan \phi_1 = a/h$.

The argument goes like this: In the notation of Figure 1.3, $m_1 = \tan \phi_1 = a/h$, while $m_2 = \tan \phi_2 = -h/a$. Hence, $m_1 m_2 = (a/h)(-h/a) = -1$.

Equations of Lines

The vertical line through the point (a, b) has equation $x = a$ since every x-coordinate on the line has the value a. Similarly, the horizontal line through (a, b) has equation $y = b$.

> **EXAMPLE 2 Finding Equations of Vertical and Horizontal Lines**
>
> The vertical and horizontal lines through the point $(2, 3)$ have equations $x = 2$ and $y = 3$, respectively (Figure 1.4).   *Now try Exercise 9.*

We can write an equation for any nonvertical line L if we know its slope m and the coordinates of one point $P_1(x_1, y_1)$ on it. If $P(x, y)$ is *any* other point on L, then

$$\frac{y - y_1}{x - x_1} = m,$$

so that

$$y - y_1 = m(x - x_1) \quad \text{or} \quad y = m(x - x_1) + y_1.$$

Figure 1.4 The standard equations for the vertical and horizontal lines through the point $(2, 3)$ are $x = 2$ and $y = 3$. (Example 2)

> **DEFINITION Point-Slope Equation**
>
> The equation
> $$y = m(x - x_1) + y_1$$
> is the **point-slope equation** of the line through the point (x_1, y_1) with slope m.
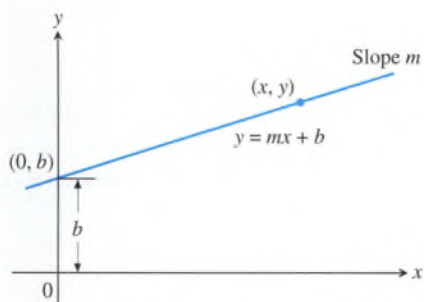
EXAMPLE 3 Using the Point-Slope Equation

Write the point-slope equation for the line through the point $(2, 3)$ with slope $-3/2$.

SOLUTION

We substitute $x_1 = 2$, $y_1 = 3$, and $m = -3/2$ into the point-slope equation and obtain

$$y = -\frac{3}{2}(x - 2) + 3 \quad \text{or} \quad y = -\frac{3}{2}x + 6.$$

Now try Exercise 13.

The y-coordinate of the point where a nonvertical line intersects the y-axis is the **y-intercept** of the line. Similarly, the x-coordinate of the point where a nonhorizontal line intersects the x-axis is the **x-intercept** of the line. A line with slope m and y-intercept b passes through $(0, b)$ (Figure 1.5), so

$$y = m(x - 0) + b, \quad \text{or, more simply,} \quad y = mx + b.$$

Figure 1.5 A line with slope m and y-intercept b.

DEFINITION **Slope-Intercept Equation**

The equation

$$y = mx + b$$

is the **slope-intercept equation** of the line with slope m and y-intercept b.

EXAMPLE 4 Writing the Slope-Intercept Equation

Write the slope-intercept equation for the line through $(-2, -1)$ and $(3, 4)$.

SOLUTION

The line's slope is

$$m = \frac{4 - (-1)}{3 - (-2)} = \frac{5}{5} = 1.$$

We can use this slope with either of the two given points in the point-slope equation. For $(x_1, y_1) = (-2, -1)$, we obtain

$$y = 1 \cdot (x - (-2)) + (-1)$$
$$y = x + 2 + (-1)$$
$$y = x + 1.$$

Now try Exercise 17.

If A and B are not both zero, the graph of the equation $Ax + By = C$ is a line. Every line has an equation in this form, even lines with undefined slopes.
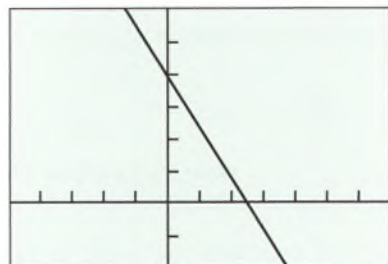

DEFINITION **General Linear Equation**

The equation

$$Ax + By = C \quad (A \text{ and } B \text{ not both } 0)$$

is a **general linear equation** in x and y.

Although the general linear form helps in the quick identification of lines, the slope-intercept form is the one to enter into a calculator for graphing.

$y = -\dfrac{8}{5}x + 4$

[−5, 7] by [−2, 6]

Figure 1.6 The line $8x + 5y = 20$. (Example 5)

EXAMPLE 5 Analyzing and Graphing a General Linear Equation

Find the slope and y-intercept of the line $8x + 5y = 20$. Graph the line.

SOLUTION

Solve the equation for y to put the equation in slope-intercept form:

$$8x + 5y = 20$$
$$5y = -8x + 20$$
$$y = -\frac{8}{5}x + 4$$

This form reveals the slope ($m = -8/5$) and y-intercept ($b = 4$), and puts the equation in a form suitable for graphing (Figure 1.6).

Now try Exercise 27.

EXAMPLE 6 Writing Equations for Lines

Write an equation for the line through the point $(-1, 2)$ that is **(a)** parallel, and **(b)** perpendicular to the line L: $y = 3x - 4$.

SOLUTION

The line L, $y = 3x - 4$, has slope 3.

(a) The line $y = 3(x + 1) + 2$, or $y = 3x + 5$, passes through the point $(-1, 2)$, and is parallel to L because it has slope 3.

(b) The line $y = (-1/3)(x + 1) + 2$, or $y = (-1/3)x + 5/3$, passes through the point $(-1, 2)$, and is perpendicular to L because it has slope $-1/3$.

Now try Exercise 31.

EXAMPLE 7 Determining a Function

The following table gives values for the linear function $f(x) = mx + b$. Determine m and b.

x	$f(x)$
−1	14/3
1	−4/3
2	−13/3

SOLUTION

The graph of f is a line. From the table we know that the following points are on the line: $(-1, 14/3)$, $(1, -4/3)$, $(2, -13/3)$.

Using the first two points, the slope m is

$$m = \frac{-4/3 - (14/3)}{1 - (-1)} = \frac{-6}{2} = -3.$$

So $f(x) = -3x + b$. Because $f(-1) = 14/3$, we have

$$f(-1) = -3(-1) + b$$
$$14/3 = 3 + b$$
$$b = 5/3.$$

continued

Thus, $m = -3$, $b = 5/3$, and $f(x) = -3x + 5/3$.

We can use either of the other two points determined by the table to check our work.

Now try Exercise 35.

Applications

Many important variables are related by linear equations. For example, the relationship between Fahrenheit temperature and Celsius temperature is linear, a fact we use to advantage in the next example.

EXAMPLE 8 Temperature Conversion

Find the relationship between Fahrenheit and Celsius temperature. Then find the Celsius equivalent of 90°F and the Fahrenheit equivalent of −5°C.

SOLUTION

Because the relationship between the two temperature scales is linear, it has the form $F = mC + b$. The freezing point of water is $F = 32°$ or $C = 0°$, while the boiling point is $F = 212°$ or $C = 100°$. Thus,

$$32 = m \cdot 0 + b \quad \text{and} \quad 212 = m \cdot 100 + b,$$

so $b = 32$ and $m = (212 - 32)/100 = 9/5$. Therefore,

$$F = \frac{9}{5}C + 32, \quad \text{or} \quad C = \frac{5}{9}(F - 32).$$

These relationships let us find equivalent temperatures. The Celsius equivalent of 90°F is

$$C = \frac{5}{9}(90 - 32) \approx 32.2°.$$

The Fahrenheit equivalent of −5°C is

$$F = \frac{9}{5}(-5) + 32 = 23°. \qquad \textit{Now try Exercise 43.}$$

It can be difficult to see patterns or trends in lists of paired numbers. For this reason, we sometimes begin by plotting the pairs (such a plot is called a **scatter plot**) to see whether the corresponding points lie close to a curve of some kind. If they do, and if we can find an equation $y = f(x)$ for the curve, then we have a formula that

1. summarizes the data with a simple expression, and

2. lets us predict values of y for other values of x.

Some graphing utilities have a feature that enables them to approximate the relationship between variables with a linear equation. We use this feature in Example 9.

The process of finding a curve to fit data is called **regression analysis** and the curve is called a **regression curve.**

There are many useful types of regression curves—power, exponential, logarithmic, sinusoidal, and so on. In the next example, we use the calculator's linear regression feature to fit the data in Table 1.1 with a line.

EXAMPLE 9 Regression Analysis—Predicting World Population

Starting with the data in Table 1.1, build a linear model for the growth of the world population. Use the model to predict the world population in the year 2010, and compare this prediction with the Statistical Abstract prediction of 6812 million.

Table 1.1 **World Population**

Year	Population (millions)
1980	4454
1985	4853
1990	5285
1995	5696
2003	6305
2004	6378
2005	6450

Source: U.S. Bureau of the Census, *Statistical Abstract of the United States, 2004-2005.*

SOLUTION

Why Not Round the Decimals in Equation 1 Even More?

If we do, our final calculation will be way off. Using $y = 80x - 153,849$, for instance, gives $y = 6951$ when $x = 2010$, as compared to $y = 6865$, an increase of 86 million. The rule is: *Retain all decimal places while working a problem. Round only at the end.* We rounded the coefficients in Equation 1 enough to make it readable, but not enough to hurt the outcome. However, we knew how much we could safely round *only from first having done the entire calculation with numbers unrounded.*

Model Upon entering the data into the grapher, we find the regression equation to be approximately

$$y = 79.957x - 153848.716, \tag{1}$$

where x represents the year and y the population *in millions.*

Figure 1.7a shows the scatter plot for Table 1.1 together with a graph of the regression line just found. You can see how well the line fits the data.
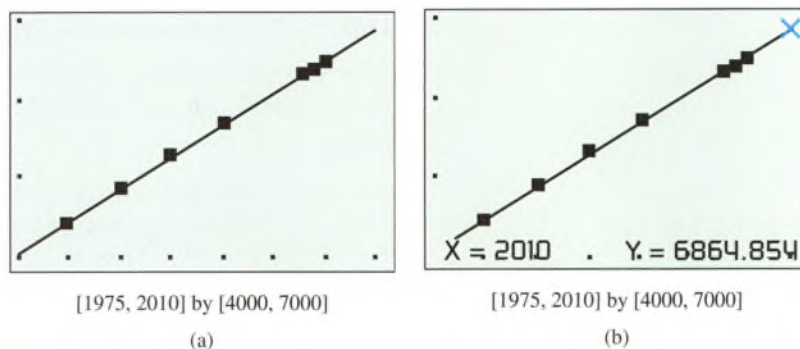

[1975, 2010] by [4000, 7000]

(a)

[1975, 2010] by [4000, 7000]

(b)

Figure 1.7 (Example 9)

Solve Graphically Our goal is to predict the population in the year 2010. Reading from the graph in Figure 1.7b, we conclude that when x is 2010, y is approximately 6865.

Rounding Rule

Round your answer as appropriate, but do not round the numbers in the calculations that lead to it.

Confirm Algebraically Evaluating Equation 1 for $x = 2010$ gives

$$y = 79.957(2010) - 153848.716$$
$$\approx 6865.$$

Interpret The linear regression equation suggests that the world population in the year 2010 will be about 6865 million, or approximately 53 million more than the Statistical Abstract prediction of 6812 million. *Now try Exercise 45.*

Regression Analysis

Regression analysis has four steps:

1. Plot the data (scatter plot).

2. Find the regression equation. For a line, it has the form $y = mx + b$.

3. Superimpose the graph of the regression equation on the scatter plot to see the fit.

4. Use the regression equation to predict y-values for particular values of x.

Quick Review 1.1   *(For help, go to Section 1.1.)*

1. Find the value of y that corresponds to $x = 3$ in
$y = -2 + 4(x - 3)$.

2. Find the value of x that corresponds to $y = 3$ in
$y = 3 - 2(x + 1)$.

In Exercises 3 and 4, find the value of m that corresponds to the values of x and y.

3. $x = 5$, $y = 2$, $m = \dfrac{y - 3}{x - 4}$

4. $x = -1$, $y = -3$, $m = \dfrac{2 - y}{3 - x}$

In Exercises 5 and 6, determine whether the ordered pair is a solution to the equation.

5. $3x - 4y = 5$
   (a) $(2, 1/4)$ **(b)** $(3, -1)$

6. $y = -2x + 5$
   (a) $(-1, 7)$ **(b)** $(-2, 1)$

In Exercises 7 and 8, find the distance between the points.

7. $(1, 0)$, $(0, 1)$

8. $(2, 1)$, $(1, -1/3)$

In Exercises 9 and 10, solve for y in terms of x.

9. $4x - 3y = 7$

10. $-2x + 5y = -3$

Section 1.1 Exercises

In Exercises 1–4, find the coordinate increments from A to B.

1. $A(1, 2)$, $B(-1, -1)$

2. $A(-3, 2)$, $B(-1, -2)$

3. $A(-3, 1)$, $B(-8, 1)$

4. $A(0, 4)$, $B(0, -2)$

In Exercises 5–8, let L be the line determined by points A and B.
   (a) Plot A and B.   **(b)** Find the slope of L.
   (c) Draw the graph of L.

5. $A(1, -2)$, $B(2, 1)$

6. $A(-2, -1)$, $B(1, -2)$

7. $A(2, 3)$, $B(-1, 3)$

8. $A(1, 2)$, $B(1, -3)$

In Exercise 9–12, write an equation for **(a)** the vertical line and **(b)** the horizontal line through the point P.

9. $P(3, 2)$

10. $P(-1, 4/3)$

11. $P(0, -\sqrt{2})$

12. $P(-\pi, 0)$

In Exercises 13–16, write the point-slope equation for the line through the point P with slope m.

13. $P(1, 1)$, $m = 1$

14. $P(-1, 1)$, $m = -1$

15. $P(0, 3)$, $m = 2$

16. $P(-4, 0)$, $m = -2$

In Exercises 17–20, write the slope-intercept equation for the line with slope m and y-intercept b.

17. $m = 3$, $b = -2$

18. $m = -1$, $b = 2$

19. $m = -1/2$, $b = -3$

20. $m = 1/3$, $b = -1$

In Exercises 21–24, write a general linear equation for the line through the two points.

21. $(0, 0)$, $(2, 3)$

22. $(1, 1)$, $(2, 1)$

23. $(-2, 0)$, $(-2, -2)$

24. $(-2, 1)$, $(2, -2)$

In Exercises 25 and 26, the line contains the origin and the point in the upper right corner of the grapher screen. Write an equation for the line.

25.

$[-10, 10]$ by $[-25, 25]$

26.

$[-5, 5]$ by $[-2, 2]$
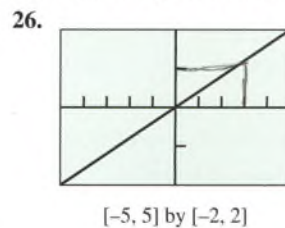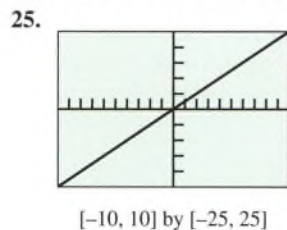
In Exercises 27–30, find the **(a)** slope and **(b)** y-intercept, and **(c)** graph the line.

27. $3x + 4y = 12$

28. $x + y = 2$

29. $\dfrac{x}{3} + \dfrac{y}{4} = 1$

30. $y = 2x + 4$

In Exercises 31–34, write an equation for the line through P that is **(a)** parallel to L, and **(b)** perpendicular to L.

31. $P(0, 0)$, $L: y = -x + 2$

32. $P(-2, 2)$, $L: 2x + y = 4$

33. $P(-2, 4)$, $L: x = 5$

34. $P(-1, 1/2)$, $L: y = 3$

In Exercises 35 and 36, a table of values is given for the linear function $f(x) = mx + b$. Determine m and b.

35.

x	$f(x)$
1	2
3	9
5	16

36.

x	$f(x)$
2	-1
4	-4
6	-7

In Exercises 37 and 38, find the value of x or y for which the line through A and B has the given slope m.

37. $A(-2, 3)$, $B(4, y)$, $m = -2/3$

38. $A(-8, -2)$, $B(x, 2)$, $m = 2$

39. *Revisiting Example 4* Show that you get the same equation in Example 4 if you use the point $(3, 4)$ to write the equation.

40. **Writing to Learn** *x- and y-intercepts*

(a) Explain why c and d are the x-intercept and y-intercept, respectively, of the line

$$\frac{x}{c} + \frac{y}{d} = 1.$$

(b) How are the x-intercept and y-intercept related to c and d in the line

$$\frac{x}{c} + \frac{y}{d} = 2?$$

41. *Parallel and Perpendicular Lines* For what value of k are the two lines $2x + ky = 3$ and $x + y = 1$ **(a)** parallel? **(b)** perpendicular?

Group Activity In Exercises 42–44, work in groups of two or three to solve the problem.
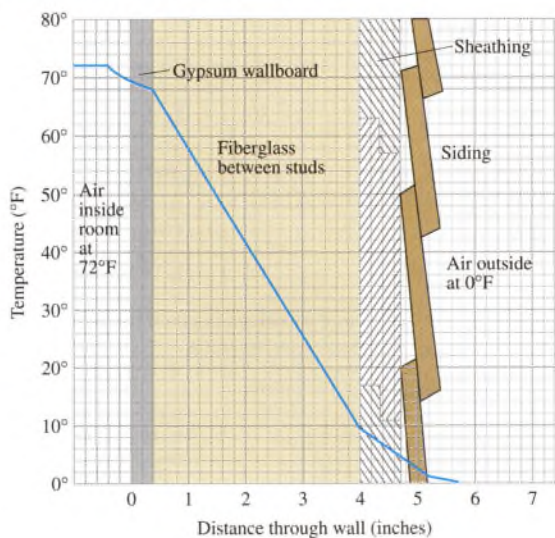
42. *Insulation* By measuring slopes in the figure below, find the temperature change in degrees per inch for the following materials.

(a) gypsum wallboard

(b) fiberglass insulation

(c) wood sheathing

(d) **Writing to Learn** Which of the materials in (a)–(c) is the best insulator? the poorest? Explain.

Distance through wall (inches)

43. *Pressure under Water* The pressure p experienced by a diver under water is related to the diver's depth d by an equation of the form $p = kd + 1$ (k a constant). When $d = 0$ meters, the pressure is 1 atmosphere. The pressure at 100 meters is 10.94 atmospheres. Find the pressure at 50 meters.

44. *Modeling Distance Traveled* A car starts from point P at time $t = 0$ and travels at 45 mph.

(a) Write an expression $d(t)$ for the distance the car travels from P.

(b) Graph $y = d(t)$.

(c) What is the slope of the graph in (b)? What does it have to do with the car?

(d) **Writing to Learn** Create a scenario in which t could have negative values.

(e) **Writing to Learn** Create a scenario in which the y-intercept of $y = d(t)$ could be 30.

In Exercises 45 and 46, use linear regression analysis.

45. Table 1.2 shows the mean annual compensation of construction workers.

Table 1.2 Construction Workers' Average Annual Compensation	
Year	Annual Total Compensation (dollars)
1999	42,598
2000	44,764
2001	47,822
2002	48,966

Source: U.S. Bureau of the Census, *Statistical Abstract of the United States, 2004–2005.*

(a) Find the linear regression equation for the data.

(b) Find the slope of the regression line. What does the slope represent?

(c) Superimpose the graph of the linear regression equation on a scatter plot of the data.

(d) Use the regression equation to predict the construction workers' average annual compensation in the year 2008.

46. Table 1.3 lists the ages and weights of nine girls.

Table 1.3 Girls' Ages and Weights	
Age (months)	Weight (pounds)
19	22
21	23
24	25
27	28
29	31
31	28
34	32
38	34
43	39

(a) Find the linear regression equation for the data.

(b) Find the slope of the regression line. What does the slope represent?

(c) Superimpose the graph of the linear regression equation on a scatter plot of the data.

(d) Use the regression equation to predict the approximate weight of a 30-month-old girl.

Standardized Test Questions

You should solve the following problems without using a graphing calculator.

47. **True or False** The slope of a vertical line is zero. Justify your answer.

48. **True or False** The slope of a line perpendicular to the line $y = mx + b$ is $1/m$. Justify your answer.

49. **Multiple Choice** Which of the following is an equation of the line through $(-3, 4)$ with slope $1/2$?

(A) $y - 4 = \dfrac{1}{2}(x + 3)$ (B) $y + 3 = \dfrac{1}{2}(x - 4)$

(C) $y - 4 = -2(x + 3)$ (D) $y - 4 = 2(x + 3)$

(E) $y + 3 = 2(x - 4)$

50. **Multiple Choice** Which of the following is an equation of the vertical line through $(-2, 4)$?

(A) $y = 4$ (B) $x = 2$ (C) $y = -4$

(D) $x = 0$ (E) $x = -2$

51. **Multiple Choice** Which of the following is the x-intercept of the line $y = 2x - 5$?

(A) $x = -5$ (B) $x = 5$ (C) $x = 0$

(D) $x = 5/2$ (E) $x = -5/2$

52. **Multiple Choice** Which of the following is an equation of the line through $(-2, -1)$ parallel to the line $y = -3x + 1$?

(A) $y = -3x + 5$ (B) $y = -3x - 7$ (C) $y = \dfrac{1}{3}x - \dfrac{1}{3}$

(D) $y = -3x + 1$ (E) $y = -3x - 4$

Extending the Ideas

53. The median price of existing single-family homes has increased consistently during the past few years. However, the data in Table 1.4 show that there have been differences in various parts of the country.

Table 1.4 **Median Price of Single-Family Homes**

Year	South (dollars)	West (dollars)
1999	145,900	173,700
2000	148,000	196,400
2001	155,400	213,600
2002	163,400	238,500
2003	168,100	260,900

Source: U.S. Bureau of the Census, *Statistical Abstract of the United States, 2004–2005.*

(a) Find the linear regression equation for home cost in the South.

(b) What does the slope of the regression line represent?

(c) Find the linear regression equation for home cost in the West.

(d) Where is the median price increasing more rapidly, in the South or the West?

54. **Fahrenheit versus Celsius** We found a relationship between Fahrenheit temperature and Celsius temperature in Example 8.

(a) Is there a temperature at which a Fahrenheit thermometer and a Celsius thermometer give the same reading? If so, what is it?

(b) **Writing to Learn** Graph $y_1 = (9/5)x + 32$, $y_2 = (5/9)(x - 32)$, and $y_3 = x$ in the same viewing window. Explain how this figure is related to the question in part (a).

55. **Parallelogram** Three different parallelograms have vertices at $(-1, 1)$, $(2, 0)$, and $(2, 3)$. Draw the three and give the coordinates of the missing vertices.

56. **Parallelogram** Show that if the midpoints of consecutive sides of any quadrilateral are connected, the result is a parallelogram.

57. **Tangent Line** Consider the circle of radius 5 centered at $(0, 0)$. Find an equation of the line tangent to the circle at the point $(3, 4)$.

58. **Group Activity** *Distance From a Point to a Line* This activity investigates how to find the distance from a point $P(a, b)$ to a line $L: Ax + By = C$.

(a) Write an equation for the line M through P perpendicular to L.

(b) Find the coordinates of the point Q in which M and L intersect.

(c) Find the distance from P to Q.