C12

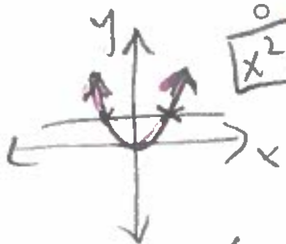
Notes:

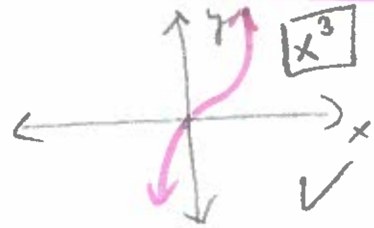
# Inverse of a Relation

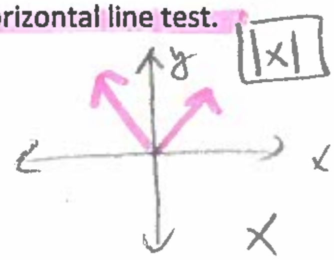
- Mapping notation: $(x, y) \rightarrow (y, x)$
  This means that the domain of the original becomes the range of the inverse and the range of the original becomes the domain of the inverse.

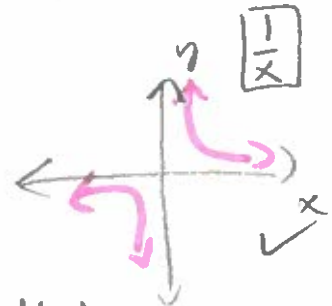
- Every relation has an inverse. Inverse of a relation is a "special reflection" of the original relation where the mirror of the reflection is the line $y = x$.

- When the inverse transformation is carried out on a function that is one-to-one then the inverse itself is also a function.
  - A one-to-one function passes the horizontal line test.










  - An inverse that is a function is denoted by: $f^{-1}(x)$

- Despite of the notation an inverse is very different from a reciprocal.

- It is possible to strategically restrict the domain of the original function so it becomes one-to-one and its inverse is then also a function. This is most commonly done with trigonometric functions.

- In general, the equation of an inverse can be found algebraically by following these steps:

  1. Replace f(x) with "y".
  2. Swap every "x" with "y" and "y" with "x".
  3. Solve for "y".
  4. Use the inverse notation if the resultant relation is also a function.

Example 1: Find the inverse of $f(x) = 0.25x - 5$

$$y = 0.25x - 5$$
$$x = 0.25y - 5$$
$$\frac{x+5}{0.25} = \frac{0.25y}{0.25}$$
$$4x + 20 = y$$

$\therefore f^{-1}(x) = 4x + 20$

$$\boxed{f^{-1}(x) = \frac{1}{m}x - \frac{b}{m}}$$

Notice: slopes are reciprocals.

* last page

Example 2: Find the inverse of $f(x) = 2x^2 + 16x - 5$. Sketch a graph of the original and of the inverse in the same coordinate system.

$$y = 2x^2 + 16x - 5$$

$$x = 2y^2 + 16y - 5 \quad \rightarrow \text{complete the square}$$

$$x = 2[y^2 + 8y] - 5$$

$$x = 2[(y+4)^2 - 16] - 5$$

$$x = 2(y+4)^2 - 37$$

not one-to-one

$$\frac{x + 37}{2} = \frac{2(y+4)^2}{2}$$

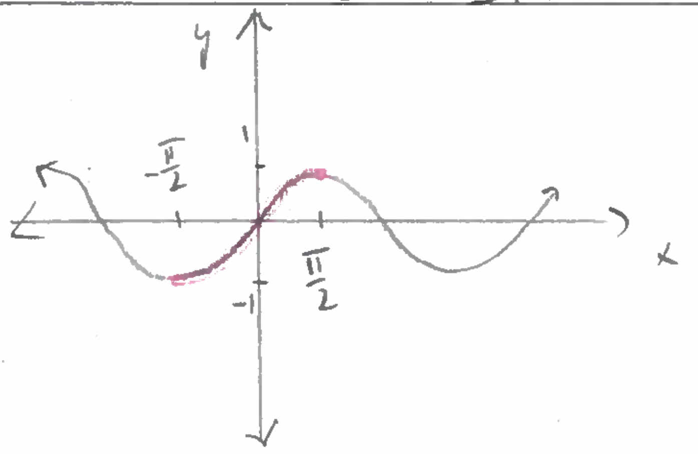
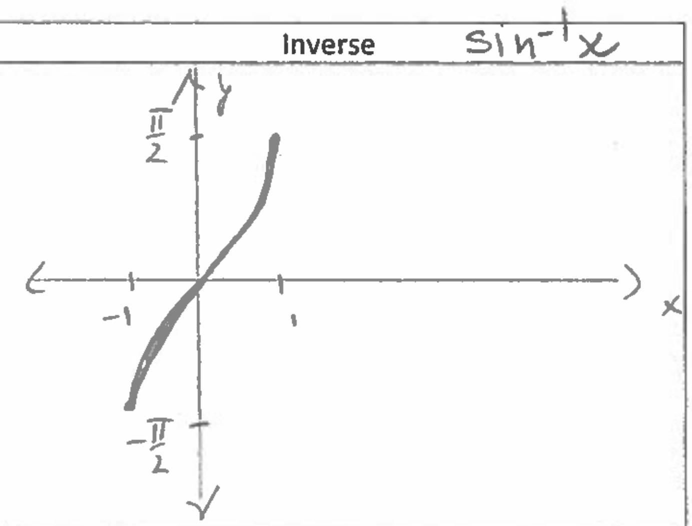
$$\sqrt{\frac{x+37}{2}} = \sqrt{(y+4)^2}$$

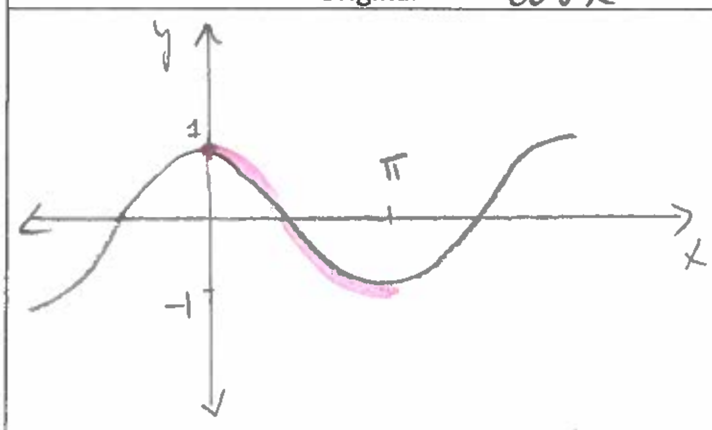
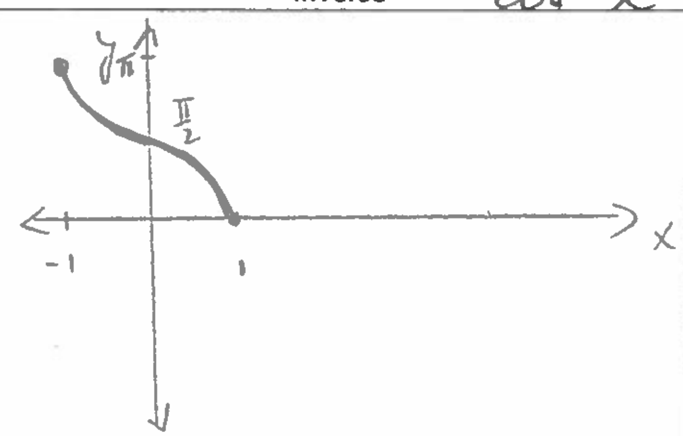
$$\pm\sqrt{\frac{x+37}{2}} - 4 = y \quad \leftarrow \text{not a function}$$

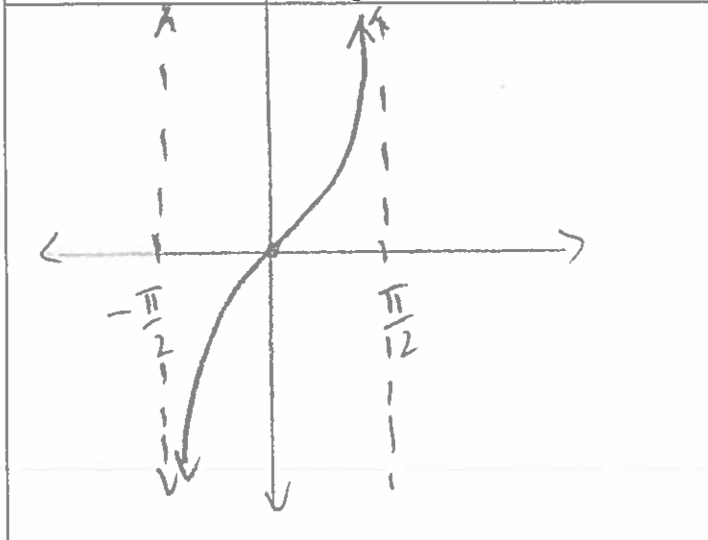
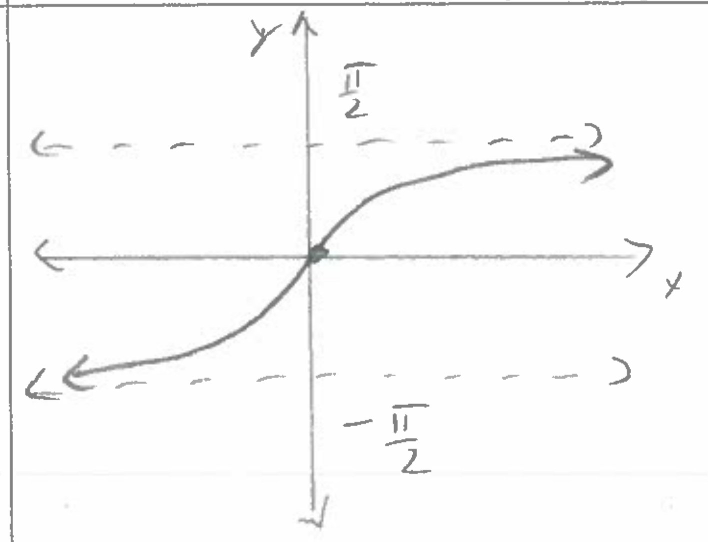
$\therefore$ the inverse of $f(x)$ is $\boxed{y = \pm\sqrt{\frac{x+37}{2}} - 4}$

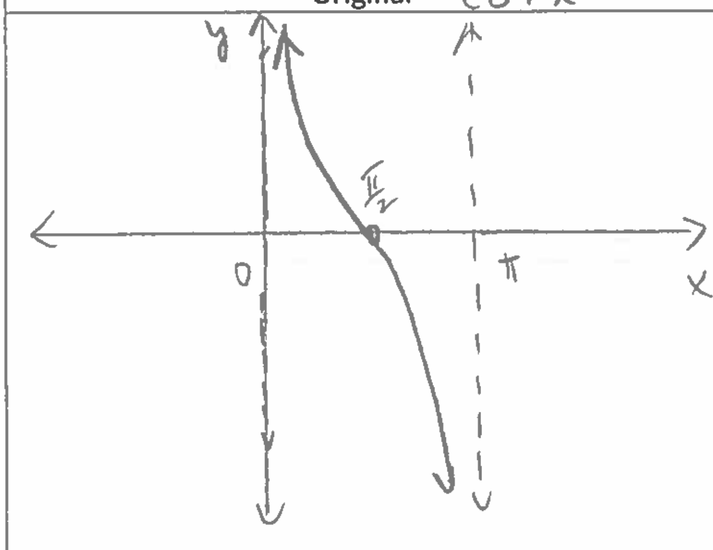
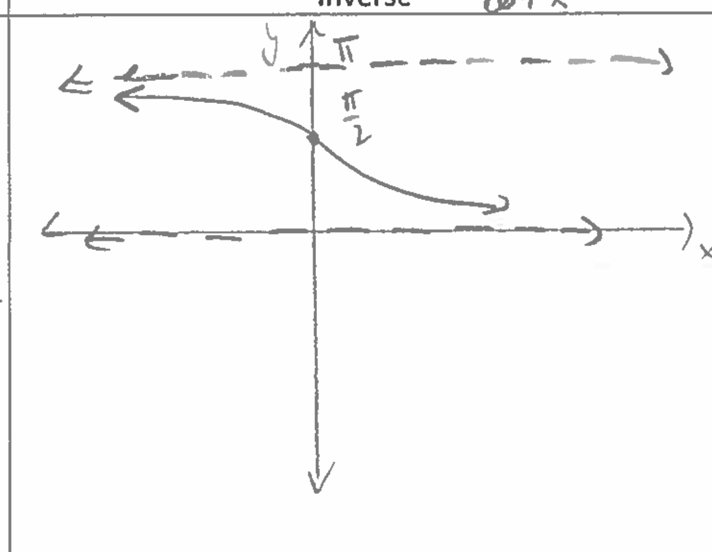
# Inverse of Trigonometric Functions

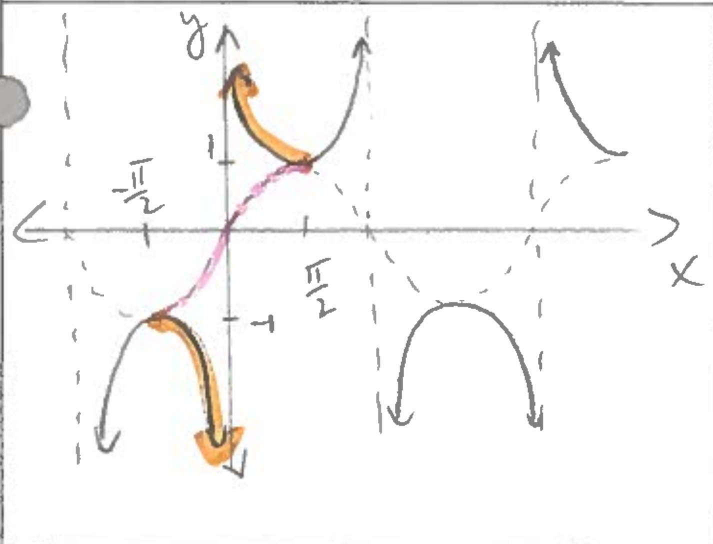
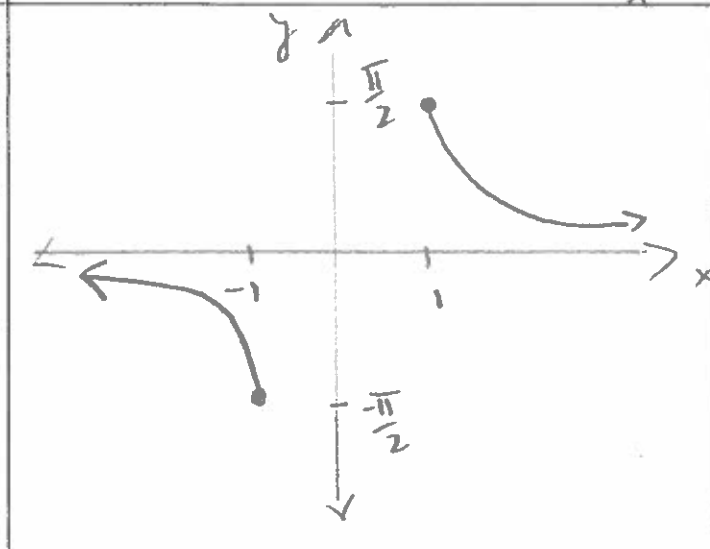
- As all trigonometric functions are periodic, they are not one-to-one.
- In order for the inverse any trigonometric function to be a function, we restrict the domain of the original in a specific manner.

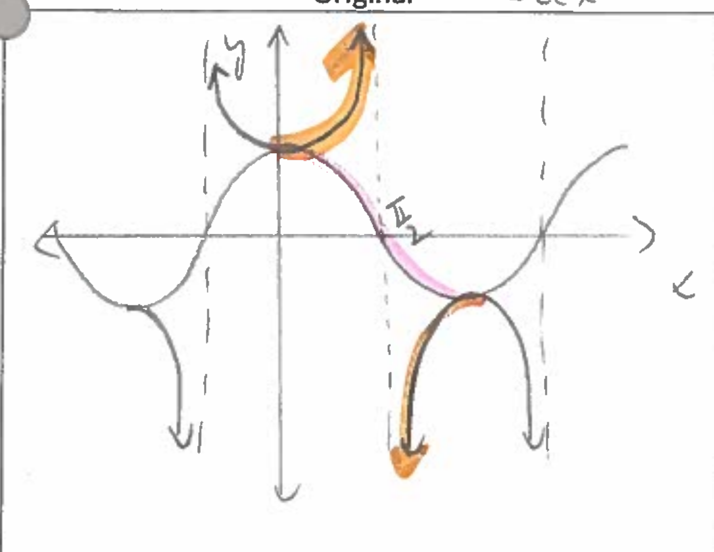
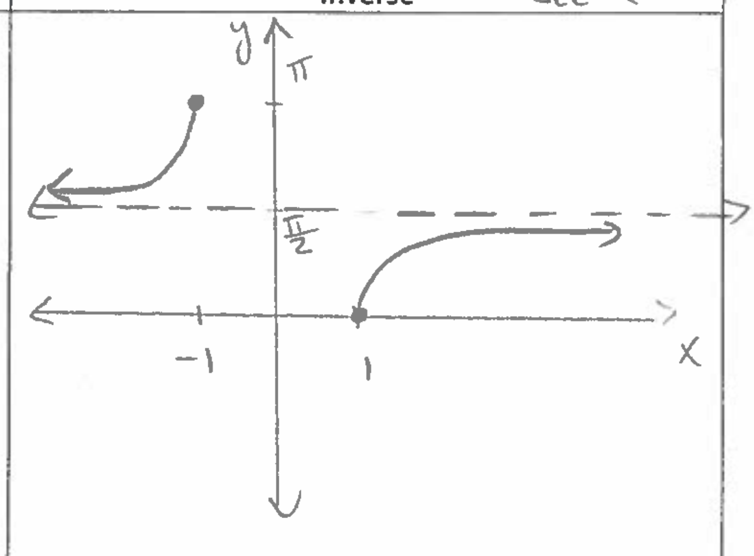
| Original $\sin x$ | | Inverse $\sin^{-1} x$ | |
|---|---|---|---|
|  | |  | |
| Domain | Restricted to $\{x \mid -\frac{\pi}{2} \le x \le \frac{\pi}{2}, x \in \mathbb{R}\}$ | Domain | $\{x \mid -1 \le x \le 1, x \in \mathbb{R}\}$ |
| Range | $\{y \mid -1 \le y \le 1, y \in \mathbb{R}\}$ | Range | $\{y \mid -\frac{\pi}{2} \le y \le \frac{\pi}{2}, y \in \mathbb{R}\}$ |

| Original $\cos x$ | | Inverse $\cos^{-1} x$ | |
|---|---|---|---|
|  | |  | |
| Domain | Restricted to $\{x \mid 0 \le x \le \pi, x \in \mathbb{R}\}$ | Domain | $\{x \mid -1 \le x \le 1, x \in \mathbb{R}\}$ |
| Range | $\{y \mid -1 \le y \le 1, y \in \mathbb{R}\}$ | Range | $\{y \mid 0 \le y \le \pi, y \in \mathbb{R}\}$ |

<table>
<tr><th colspan="2">Original $\tan x$</th><th colspan="2">Inverse $\tan^{-1} x$</th></tr>
<tr><td colspan="2">
</td><td colspan="2">
</td></tr>
<tr><td>Domain</td><td>Restricted to $\{x \mid -\frac{\pi}{2} < x < \frac{\pi}{2}, x \in \mathbb{R}\}$</td><td>Domain</td><td>$\{x \mid x \in \mathbb{R}\}$</td></tr>
<tr><td>Range</td><td>$\{y \mid y \in \mathbb{R}\}$</td><td>Range</td><td>$\{y \mid -\frac{\pi}{2} < y \frac{\pi}{2}, y \in \mathbb{R}\}$</td></tr>
</table>

<table>
<tr><th colspan="2">Original $\cot x$</th><th colspan="2">Inverse $\cot^{-1} x$</th></tr>
<tr><td colspan="2">
</td><td colspan="2">
</td></tr>
<tr><td>Domain</td><td>Restricted to $\{x \mid 0 < x < \pi, x \in \mathbb{R}\}$</td><td>Domain</td><td>$\{x \mid x \in \mathbb{R}\}$</td></tr>
<tr><td>Range</td><td>$\{y \mid y \in \mathbb{R}\}$</td><td>Range</td><td>$\{y \mid 0 < y < \pi, y \in \mathbb{R}\}$</td></tr>
</table>

| Original $\csc x$ | | Inverse $\csc^{-1} x$ | |
|---|---|---|---|
|   | |   | |
| Domain | Restricted to $\{x \mid -\frac{\pi}{2} \le x \le \frac{\pi}{2} \cap x \ne 0, x \in \mathbb{R}\}$ | Domain | $\{x \mid x \le -1 \cup x \ge -1, x \in \mathbb{R}\}$ |
| Range | $\{y \mid y \le -1 \cup y \ge 1, y \in \mathbb{R}\}$ | Range | $\{y \mid -\frac{\pi}{2} \le y \le \frac{\pi}{2} \cap y \ne 0, y \in \mathbb{R}\}$ |

| Original $\sec x$ | | Inverse $\sec^{-1} x$ | |
|---|---|---|---|
|   | |   | |
| Domain | Restricted to $\{x \mid 0 \le x \le \pi \cap x \ne \frac{\pi}{2}, x \in \mathbb{R}\}$ | Domain | $\{x \mid x \le -1 \cup x \ge 1, x \in \mathbb{R}\}$ |
| Range | $\{y \mid y \le -1 \cup y \ge 1, y \in \mathbb{R}\}$ | Range | $\{y \mid 0 \le y \le \pi \cap y \ne \frac{\pi}{2}, y \in \mathbb{R}\}$ |

Example 2 sketch: