Recall: a critical point $\begin{cases} f'(x) = 0 \\ f'(x) \text{ does not exist} \end{cases}$

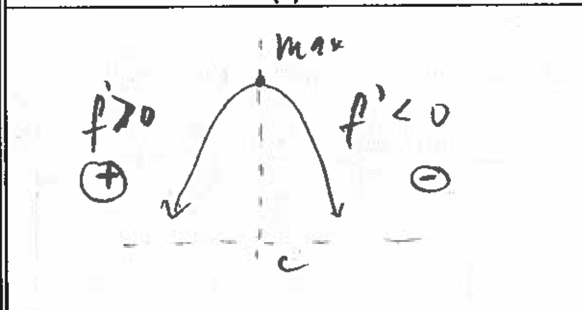
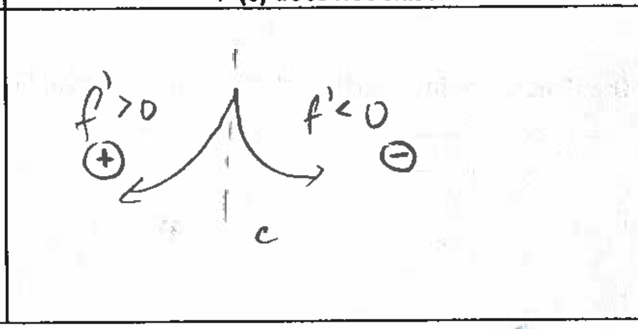
CALCULUS 12 **CONNECTING f' and f'' TO THE GRAPH OF f**

THEOREM: FIRST DERIVATIVE TEST FOR LOCAL EXTREMA

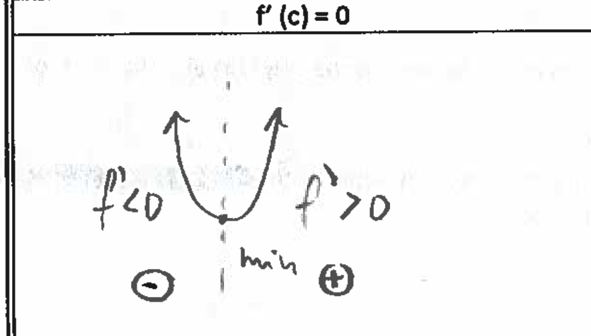
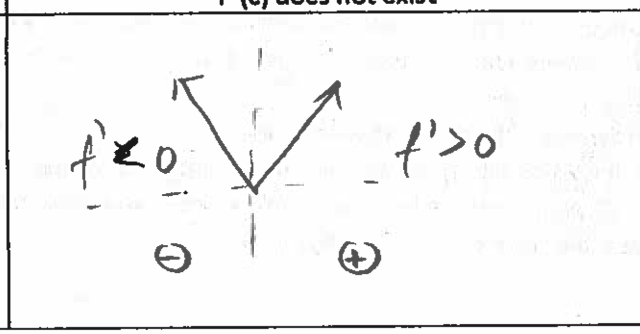
The following test applies to a continuous function f(x).

At a critical point c:

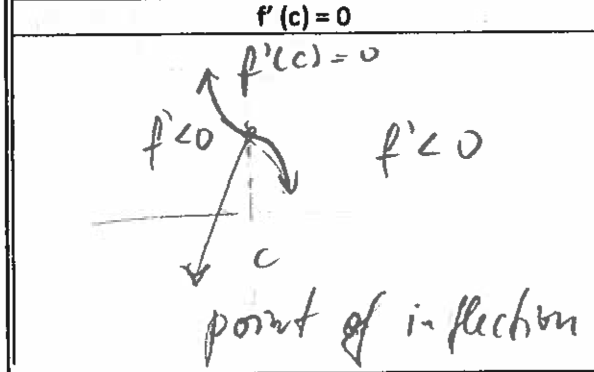
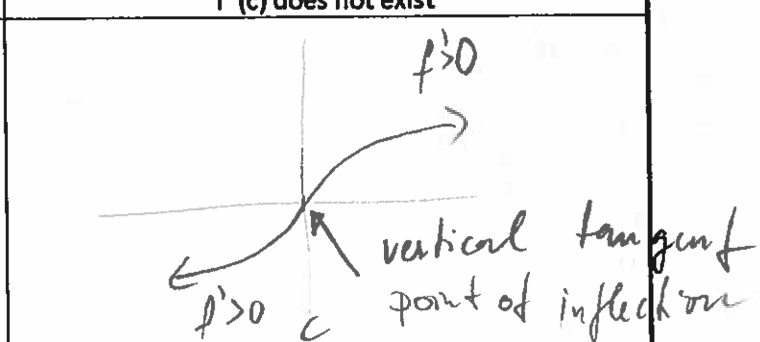
1. If f' changes sign from positive to negative at c (f'>0 for all x<c and f'<0 for all x>c), then f has a local **maximum** at c.

f' (c) = 0	f' (c) does not exist
	

2. If f' changes sign from negative to positive at c (f'< cfor all x<c and f'> 0 for all x>c), then f has a local **minimum** at c.

f' (c) = 0	f' (c) does not exist
	

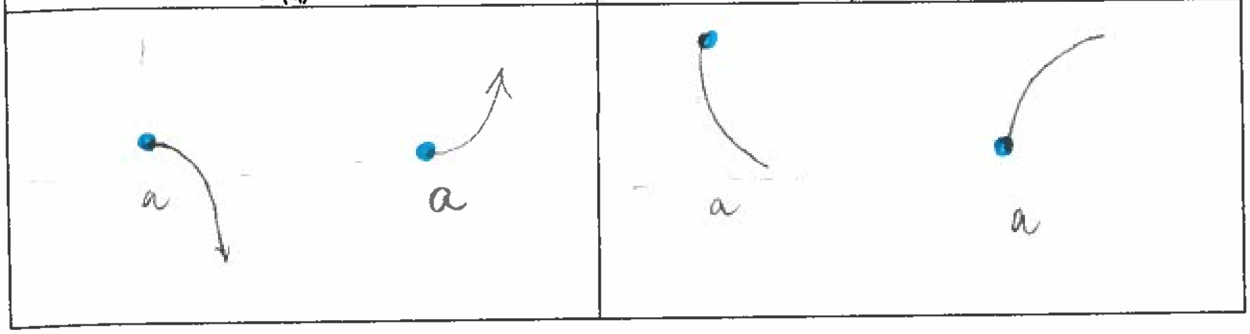
3. If f' does not change sign at c (f' has the same sign on both sides of c), then f has no local extreme value at c.

f' (c) = 0	f' (c) does not exist
	

$\lim_{h \to 0} \frac{f(a+h)-f(a)}{h} = \pm\infty$

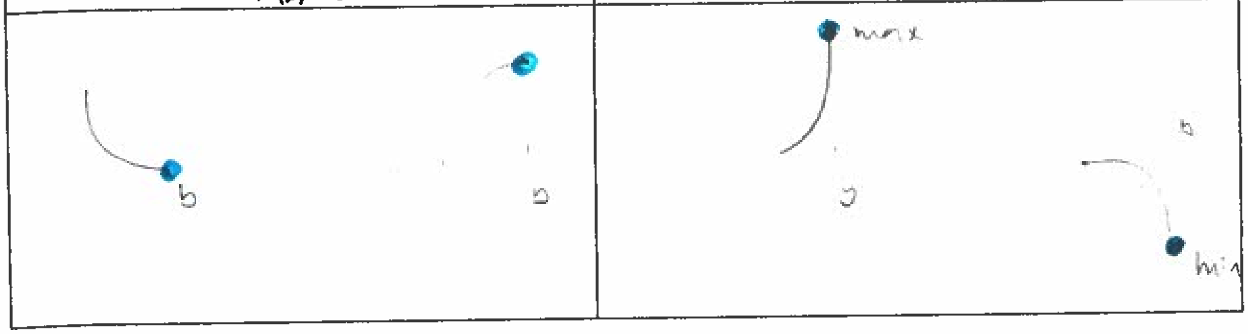
- **At the left endpoint a:** if f'<0 (f'>0) for all x> a, then f has a local maximum (minimum) value at a.

f' (a) = 0	f' (a) does not exist



- **At the right endpoint b :** if f'<0 (f'>0) for all x< b, then f has a local minimum (maximum) value at b.

f' (b) = 0	f' (b) does not exist



Example: Use the first derivative test to identify all local extrema of $y = -x^3+12x-5$

Note: y is a polynomial function that is continuous and differentiable on its entire domain. So the only critical points are those where y has a horizontal tangent.

Step 1:Find dy/dx and the points where it is zero. = critical points

Step 2: Use these points to split a number line into intervals. Within each interval identify whether the dy/dx is positive or negative.

Step 3: Identify whether the critical points yield a local maximum or minimum.

Step 4. Sketch the graph of y to check your work.

$y' = -3x^2 + 12$

$0 = -3x^2 + 12$

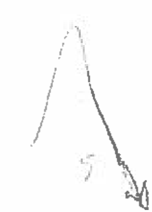
$-12 = -3x^2$

$x^2 = 4$

$x = \pm\sqrt{4}$

$x = \pm 2$

f'<0 ⊖ ↘ | -2 ↑ local minimum | f'>0 ⊕ ↗ | local max ↓ 2 | ⊖ f'<0 ↘ — f'(x), f(x)



$y'(-3) = -3(-3)^2 + 12 = -27 + 12 = \underline{-15}$

$y'(0) = \underline{12}$

$y'(3) = -3(3)^2 + 12 = \underline{-15}$

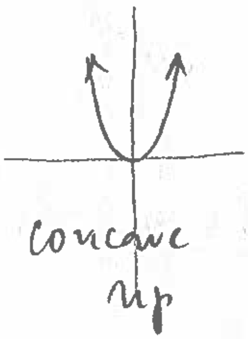
CONCAVITY

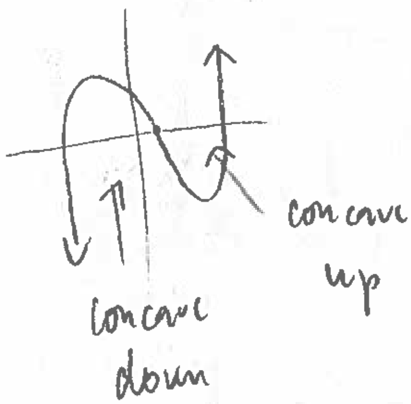
DEFINTION: CONCAVITY

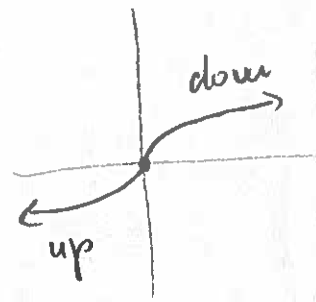
The graph of a differentiable function y = f(x) is

1. **Concave up** on an open interval I if y' is increasing on I.
2. **Concave down** on an open interval I if y' is decreasing on I.

Examples:







➤ **Concavity Test: The graph of a twice-differentiable function y = f(x) is**

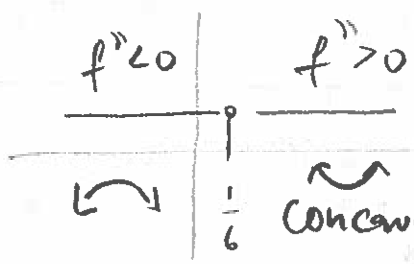
1. concave up on any interval where y'' > 0

2. concave down on any interval where y'' < 0

Example: Identify intervals on which $y = 2x^3 - x^2 - 5x$ is concave up and concave down.

$y' = 6x^2 - 2x - 5$

$y'' = 12x - 2$



Concave up: $y'' > 0$ when $12x - 2 > 0$

$12x > 2$

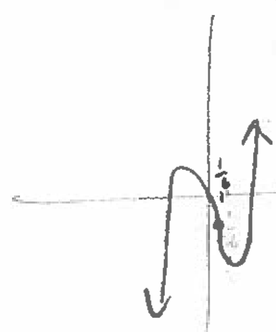
$\boxed{x > \frac{1}{6}}$

Concave down: $y'' < 0$

$12x - 2 < 0$

$12x < 2$

$\boxed{x < \frac{1}{6}}$



$\therefore$ y is concave up on $(\frac{1}{6}, \infty)$

$\therefore$ y is concave down: $(-\infty, \frac{1}{6})$

POINTS OF INFLECTION

DEFINITION: a point where the graph of a function has a tangent line and where the concavity changes, is a point of inflection.

- **A point of inflection can be either at the point where the second derivative zero or at the point where the second derivative does not exist.**
- **Not all points that have a zero second derivative are inflection points.**
- **An inflection point implies the change in the sign of the second derivative.**

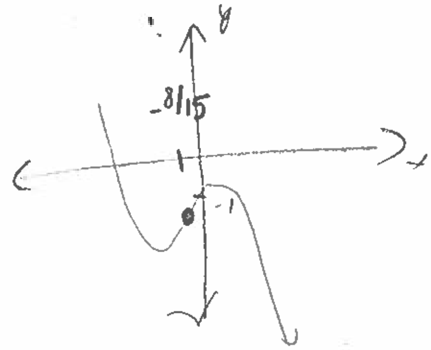
Examples:

$f'(0)$ = DNE

$f''(0)$ = DNE

but inflection

concavity changes



Inflection point

⊕ ⊖ y''

$-8/15$ y

$y = -5x^3 - 8x^2 - 1$

$y' = -15x^2 - 16x$

$y'' = -30x - 16$

• concave up : $y'' > 0$
$-30x - 16 > 0$
$x < \frac{16}{30} = -8/15$
Flip

• concave down : $y'' < 0$
$-30x - 16 < 0$
$x > -8/15$
Flip

SECOND DERIVATIVE TEST FOR LOCAL EXTREMA

1. If f'(c) = 0 and f''(c)< 0, then f has a local maximum at x = c.

2. If f'(c) = 0 and f''(c)> 0, then f has a local minimum at x = c.

Note: this test fails if f''(c) = 0 or when f''(c) fails to exists and the first derivative test must be used instead.

$f(x) = 2x^3 - 18x + 5$

$f'(x) = 6x^2 - 18$

$f''(x) = 12x$

$f'(x) = 0 = 6x^2 - 18$

$18 = 6x^2$

$x^2 = 3$

$x = \pm\sqrt{3}$

local max

$-\sqrt{3}$ $\sqrt{3}$

$f''(\sqrt{3}) > 0$

$f''(-\sqrt{3}) < 0$

local min

Copy the chart from p 213:

Hw: p 215 #1-24