Notes:

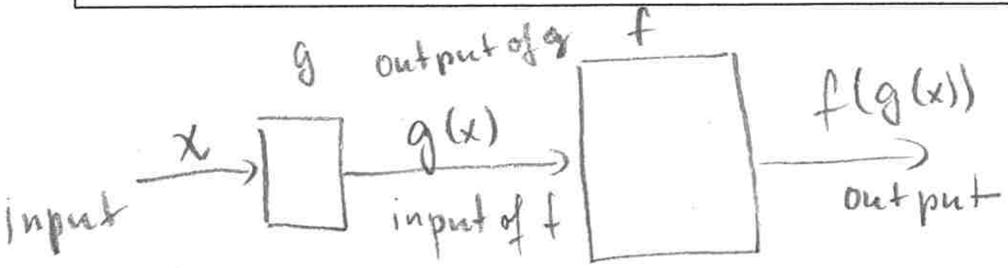
C12

Composite Function

If f and g are functions, the composite function $f \circ g$ ("f composed with g") is defined by

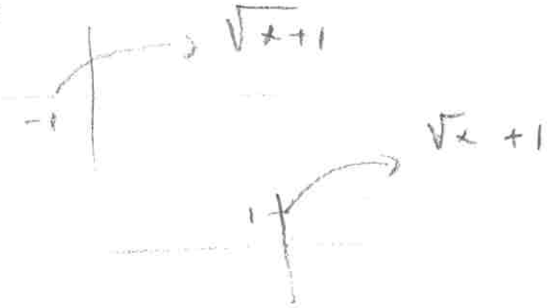
$$(f \circ g)(x) = f(g(x)).$$

The domain of the composite function consists of numbers x in the domain of function g for which $g(x)$ lies in the domain of function f.



▷ the output of the inner function becomes the input of the outer function.

Example 1: If $f(x) = \sqrt{x}$ and $g(x) = x + 1$ find the following:



a) $(f \circ g)(x) = \sqrt{x+1}$

b) $(g \circ f)(x) = \sqrt{x} + 1$

c) $(f \circ f)(x) = \sqrt{\sqrt{x}} = \left(x^{\frac{1}{2}}\right)^{\frac{1}{2}} = x^{\frac{1}{4}} = \sqrt[4]{x}$

d) $(g \circ g)(x) = x+1 + 1$

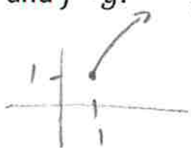
$= \boxed{x+2}$

Operations with Function

Operation	Notation	Alternative Notation	Domain	Range
Addition	$f(x)+g(x)$	$(f+g)(x)$	$D_f \cap D_g$	
Subtraction	$f(x)-g(x)$	$(f-g)(x)$	$D_f \cap D_g$	best determined from a graph
Multiplication	$f(x)\cdot g(x)$	$(f\cdot g)(x)$	$D_f \cap D_g$	
Division	$\frac{f(x)}{g(x)}$	$\left(\frac{f}{g}\right)(x)$	$D_f \cap D_g \cap g(x) \neq 0$	
Composition	$f(g(x))$	$(f \circ g)(x)$	$D_g \cap D_{f(g(x))}$	

Example 2: Find the domains and ranges of $f, g, f+g$ and $f\cdot g$.

a) $f(x) = x$ and $g(x) = \sqrt{x-1}$



$(f+g)(x) = x + \sqrt{x-1}$

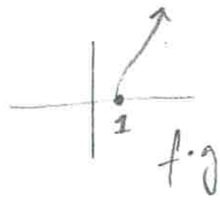
$(f\cdot g)(x) = x\sqrt{x-1} = \sqrt{x^2(x-1)}$

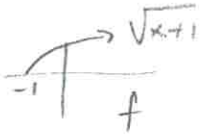
$D_f : \{x \mid x \in \mathbb{R}\}$ $\quad R_f : \{y \mid y \in \mathbb{R}\}$

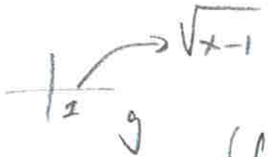
$D_g : \{x \mid x \geq 1, x \in \mathbb{R}\}$ $\quad R_g : \{y \mid y \geq 0, y \in \mathbb{R}\}$

$D_{f+g} : \{x \mid x \geq 1, x \in \mathbb{R}\}$ $\quad R_{f+g} : \{y \mid y \geq 1, y \in \mathbb{R}\}$

$D_{f\cdot g} : \{x \mid x \geq 1, x \in \mathbb{R}\}$ $\quad R_{f\cdot g} : \{y \mid y \geq 0, y \in \mathbb{R}\}$

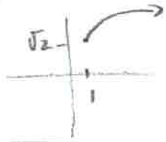






b) $f(x) = \sqrt{x+1}$, $g(x) = \sqrt{x-1}$

$(f+g)(x) = \sqrt{x+1} + \sqrt{x-1}$



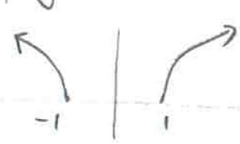
$D_f : \{x \mid x \geq -1, x \in \mathbb{R}\}$ $\quad R_f : \{y \mid y \geq 0, y \in \mathbb{R}\}$

$D_g : \{x \mid x \geq 1, x \in \mathbb{R}\}$ $\quad R_g : \{y \mid y \geq 0, y \in \mathbb{R}\}$

$D_{f+g} : \{x \mid x \geq 1, x \in \mathbb{R}\}$ $\quad R_{f+g} : \{y \mid y \geq \sqrt{2}, y \in \mathbb{R}\}$

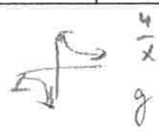
$(f\cdot g)(x) = \sqrt{x^2-1}$

$D_{f\cdot g} : \{x \mid x \leq -1 \cup x \geq 1, x \in \mathbb{R}\}$ $\quad R_{f\cdot g} : \{y \mid y \geq 0, y \in \mathbb{R}\}$



Example 3: If $f(x) = x + 5$ and $g(x) = x^2 - 3$, find the following:

$f(g(0))$	$0^2-3+5 = \underline{2}$	$g(f(0))$	$(0+5)^2-3 = \underline{22}$
$f(g(x))$	$x^2-3+5 = \underline{x^2+2}$	$g(f(x))$	$\underline{(x+5)^2-3}$ OR $\underline{x^2+10x+22}$
$f(f(-5))$	$-5+5+5 = \underline{5}$	$g(g(2))$	$(2^2-3)^2-3 = \underline{-2}$
$f(f(x))$	$x+5+5 = \underline{x+10}$	$g(g(x))$	$(x^2-3)^2-3 = \underline{x^4-6x^2+6}$



Example 4: If $f(x) = \sqrt{x}$ and $g(x) = \frac{4}{x}$, $h(x) = 4x - 8$, find the following:

$h(g(f(x)))$	$4\left(\frac{4}{\sqrt{x}}\right)-8$ $\frac{16}{\sqrt{x}}-8 = \underline{16x^{-\frac{1}{2}}-8}$	$g(f(h(x)))$	$\frac{4}{\sqrt{4x-8}}$ $= 4(4x-8)^{-}$ $= 4(4(x-2)$ $\frac{4}{2}(x$ $\boxed{2(x-}$
$f(h(g((x)))$	$\sqrt{4\left(\frac{4}{x}\right)-8}$ $\sqrt{\frac{16}{x}-8}$	$f(g(h(x)))$	$\sqrt{\frac{4}{4x-8}}$

$$= \sqrt{\frac{16-8x}{x}}$$

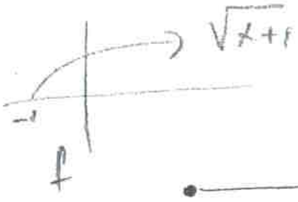
$$= \underline{(16x^{-1}-8)^{\frac{1}{2}}}$$

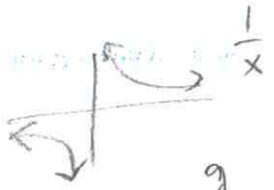
$$= \frac{\sqrt{4}}{\sqrt{4x-8}}$$

$$= \underline{2(4x-8)^{-\frac{1}{2}}}$$

$$= 2[4(x-2)]^{-\frac{1}{2}}$$

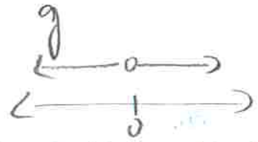
$$= 2 \cdot \frac{1}{2}(x-2)^{-\frac{1}{2}}) = \frac{1}{\sqrt{x-2}}$$





D_f :

D_g :



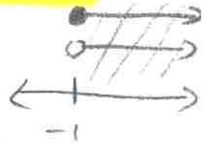
Example 5: Write the formula for f(g(x)) and g(f(x)) and find the domain and range of each:

a) $f(x) = \sqrt{x+1}$ and $g(x) = \frac{1}{x}$.

$g(f(x)) = \frac{1}{\sqrt{x+1}}$

$D_{g \circ f} : D_f \cap D_{g(f(x))}$ $\{x \mid x > -1, x \in \mathbb{R}\}$

$x+1 \geq 0$, $x \geq -1$ $\quad$ $\sqrt{x+1} \neq 0$ so $x+1 > 0$, $x > -1$



$R_{g \circ f} : \{y \mid y > 0, y \in \mathbb{R}\}$

$f(g(x)) = \sqrt{\frac{1}{x} + 1}$

$= \sqrt{\frac{1+x}{x}}$

$= \left(\frac{1+x}{x}\right)^{\frac{1}{2}}$

$D_{f \circ g} : D_g \cap D_{f(g(x))}$ = combining $x \neq 0$, case I and case II will give

$x \neq 0$ $\quad$ $g(x) \geq -1$

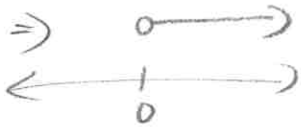
$D : \{x \mid x > 0 \cup x \leq -1, x \in \mathbb{R}\}$

$R : \{y \mid y \neq 1, y \in \mathbb{R}\}$

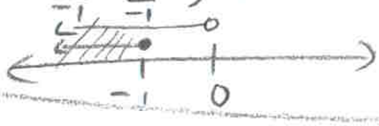
$g(x) \geq -1$

$\frac{1}{x} \geq -1$

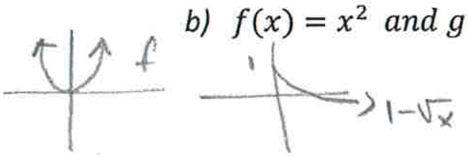
I. $x > 0 \Rightarrow$ a reciprocal of any positive number will be greater than -1 $\Rightarrow$



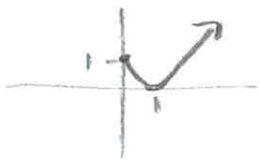
II. $x < 0 \cap \frac{1}{x} \geq -1 \rightarrow \frac{1}{-1} \leq \frac{-x}{-1} \Rightarrow -1 \geq x$



b) $f(x) = x^2$ and $g(x) = 1 - \sqrt{x}$



$f(g(x)) = (1-\sqrt{x})^2$



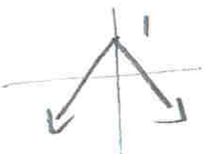
$D_{f \circ g}$ $\{x \mid x \geq 0, x \in \mathbb{R}\}$

$R_{f \circ g}$ $\{y \mid y \geq 0, y \in \mathbb{R}\}$

$g(f(x)) = 1 - \sqrt{x^2}$

! $= 1 - |x|$

$= -|x| + 1$



$D_{g \circ f} : \{x \mid x \in \mathbb{R}\}$

$R_{g \circ f} : \{y \mid y \leq 1, y \in \mathbb{R}\}$