Notes:

BINOMIAL THEOREM

Binomial theorem is a method for expanding an expression in the form $(x+y)^n$, where n is a positive integer.

Two methods can be used:

1. **Pascal's Triangle**
2. **Binomial Expansion**

Binomial Expansion:

$$(a+b)^n = \binom{n}{0} a^n b^0 + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 + \ldots + \binom{n}{n-2} a^2 b^{n-2} + \binom{n}{n-1} a^1 b^{n-1} + \binom{n}{n} a^0 b^n$$

- Each $\binom{n}{r}$ comes from a combination formula ${}_nC_r = \frac{n!}{r!(n-r)!}$ and gives you coefficients for each term. These coefficients are referred to as binomial coefficients.

Example1: Expand $(m+2)^4$

1. n=4, a=m, b=2

2. Replace a, b and n in the formula with the given values.

$$(m+2)^4 = \binom{4}{0} m^4 (2)^0 + \binom{4}{1} m^{4-1} (2)^1 + \binom{4}{2} m^{4-2} (2)^2 + \binom{4}{3} m^1 (2)^{4-1} + \binom{4}{4} m^0 (2)^4$$

<u>Notice the following:</u>

- Exponents of m start at 4 and end in 0
- Exponents of 2 begin at 0 and end in 4
- Sum of exponents in each term is always 4

3. Find binomial coefficients

$\binom{4}{0}$ =1

$\binom{4}{1}$ =4

$\binom{4}{2}$ =6 ------ notice that we get a mirror image once we reach the middle

$\binom{4}{3}$ =4

$\binom{4}{4}$ =1

4. Replace all $\binom{n}{r}$ with the coefficients found in step 3.

$=1(m)^4(2)^0 + 4(m)^3(2)^1 + 6(m)^2(2)^2 + 4(m)^1(2)^3 + 1(m)^0(2)^4$

5. Raise the monomials to the powers specified for each term

$= 1(m)^4(1) + 4(m)^3(2) + 6(m)^2(4) + 4(m)^1(8) + 1(1)(16)$

6. Combine and simplify:

$= m^4 + 8m^3 + 24m^2 + 32m + 16$

Note:

1. Binomial coefficient $\binom{n}{r}$ will not necessarily be the coefficient in your final answer

2. The Binomial theorem is written as sum of two monomials, so if your task is to expand the difference (as below in example 2), your answer will have alternating positive and negative signs.

3. The exponent of the first monomial begins at n and decreases by 1 for each term until it reaches 0.

4. The exponent for the second monomial begins at 0 and increases by 1 until it reaches n at the last term.

5. Exponents of both monomials should add to n in each term, unless the monomials themselves have powers greater than 1.

6. Binomial coefficients can be found using $\binom{n}{r}$ = combination formula or Pascal's Triangle if the exponent is relatively small.

Example 2: Expand $(2y-1)^4$

1. n=4, a=2y, b= -1

2. Replace a, b and n in the formula with the given values.

$$(2y-1)^4 = \binom{4}{0}(2y)^4(-1)^0 + \binom{4}{1}(2y)^{4-1}(-1)^1 + \binom{4}{2}(2y)^{4-2}(-1)^2 + \binom{4}{3}(2y)^1(-1)^{4-1} + \binom{4}{4}(2y)^0(-1)^4$$

3. Find the binomial coefficients by evaluating the combination formula or by Pascal's Triangle

$\binom{4}{0}=1$

$\binom{4}{1}=4$

$\binom{4}{2}=6$

$\binom{4}{3}=4$

$\binom{4}{4}=1$

4. Replace all $\binom{n}{r}$ with the coefficients found in step 3.

$=1(2y)^4(-1)^0 + 4(2y)^3(-1)^1 + 6(2y)^2(-1)^2 + 4(2y)^1(-1)^3 + 1(2y)^0(-1)^4$

5. Raise the monomials to the powers specified for each term

$= 1(2y)^4(1) + 4(2y)^3(-1) + 6(2y)^2(1) + 4(2y)^1(-1) + 1(1)(1)$

6. Combine and simplify

$= 16y^4 - 32y^3 + 24y^2 - 8y + 1$

Example 3: When monomials are raised to a power before you begin: Expand the expression $(3x^2-2y)^7$

1. a=3x², b=-2y, n=7

$\therefore (3x^2-2y)^7 = \binom{7}{0}(3x^2)^7(-2y)^0 + \binom{7}{1}(3x^2)^{7-1}(-2y)^1 + \binom{7}{2}(3x^2)^{7-2}(-2y)^2 + \binom{7}{3}(3x^2)^{7-3}(-2y)^3 + \binom{7}{4}(3x^2)^{7-4}(-2y)^4 + \binom{7}{5}(3x^2)^{7-5}(-2y)^5 + \binom{7}{6}(3x^2)^{7-6}(-2y)^6 + \binom{7}{7}(3x^2)^{7-7}(-2y)^7$

3.

$\binom{7}{0}=1$

$\binom{7}{1}=7$

$\binom{7}{2}=21$

$\binom{7}{3}=35$

$\binom{7}{4}=35$

$\binom{7}{5}=21$

$\binom{7}{6}=7$

$\binom{7}{7}=1$

4.

$=1(3x^2)^7(-2y)^0 + 7(3x^2)^{7-1}(-2y)^1 + 21(3x^2)^{7-2}(-2y)^2 + 35(3x^2)^{7-3}(-2y)^3 + 35(3x^2)^{7-4}(-2y)^4 + 21(3x^2)^{7-5}(-2y)^5 + 7(3x^2)^{7-6}(-2y)^6 + 1(3x^2)^{7-7}(-2y)^7$

5.

$=1(2187x^{14})(1) + 7(729x^{12})(-2y) + 21(243x^{10})(4y^2) + 35(81x^8)(-8y^3) + 35(27x^6)(16y^4) + 21(9x^4)(-32y^5) + 7(3x^2)(64y^6) + 1(1)(-128y^7)$

6.

$= 2187x^{14} - 10206x^{12}y + 20412x^{10}y^2 - 22680x^8y^3 + 15120x^6y^4 - 6048x^4y^5 + 1344x^2y^6 - 128y^7$

Practice:

1. Expand $(x+3y)^3$

$= \binom{3}{0} x^3 \cdot (3y)^0 + \binom{3}{1} x^2 \cdot (3y)^1 + \binom{3}{2} x^1 \cdot (3y)^2 + \binom{3}{3} x^0 \cdot (3y)^3$

$= x^3 + 3x^2 \cdot 3y + 3x \cdot 9y^2 + 1 \cdot 27y^3$

$= x^3 + 9x^2y + 27xy^2 + 27y^3$

2. Expand $(2y^2-3)^5$

$= \binom{5}{0}(2y^2)^5(-3)^0 + \binom{5}{1}(2y^2)^4 \cdot (-3)^1 + \binom{5}{2}(2y^2)^3 \cdot (-3)^2 + \binom{5}{3}(2y^2)^2 \cdot (-3)^3 +$

$+ \binom{5}{4}(2y^2)^1 \cdot (-3)^4 + \binom{5}{5}(2y^2)^0(-3)^5$

$= 32y^{10} - 5(16)y^8(3) + 10(8)y^6(9) + 10(4y^4)(-27) + 5(2y^2)(81) +$

$+ 1(1)(-243)$

$= 32y^{10} - 240y^8 + 720y^6 - 1080y^4 + 810y^2 - 243$