The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

Euclid Contest

Wednesday, April 7, 2021
(in North America and South America)

Thursday, April 8, 2021
(outside of North America and South America)

UNIVERSITY OF
WATERLOO

Time: $2\frac{1}{2}$ hours

Do not open this booklet until instructed to do so.

Number of questions: 10 **Each question is worth 10 marks**

Calculating devices are allowed, provided that they do not have any of the following features: (i) internet access, (ii) the ability to communicate with other devices, (iii) information previously stored by students (such as formulas, programs, notes, etc.), (iv) a computer algebra system, (v) dynamic geometry software.

Parts of each question can be of two types:

1. **SHORT ANSWER** parts indicated by
   - worth 3 marks each
   - full marks given for a correct answer which is placed in the box
   - **part marks awarded only if relevant work** is shown in the space provided
2. **FULL SOLUTION** parts indicated by
   - worth the remainder of the 10 marks for the question
   - **must be written in the appropriate location** in the answer booklet
   - marks awarded for completeness, clarity, and style of presentation
   - a correct solution poorly presented will not earn full marks

WRITE ALL ANSWERS IN THE ANSWER BOOKLET PROVIDED.

- Extra paper for your finished solutions supplied by your supervising teacher must be inserted into your answer booklet. Write your name, school name, and question number on any inserted pages.
- Express answers as simplified exact numbers except where otherwise indicated. For example, $\pi + 1$ and $1 - \sqrt{2}$ are simplified exact numbers.

Do not discuss the problems or solutions from this contest online for the next 48 hours.

The name, grade, school and location, and score range of some top-scoring students will be published on our website, cemc.uwaterloo.ca. In addition, the name, grade, school and location, and score of some top-scoring students may be shared with other mathematical organizations for other recognition opportunities.

NOTE:

1. Please read the instructions on the front cover of this booklet.
2. Write all answers in the answer booklet provided.
3. For questions marked , place your answer in the appropriate box in the answer booklet and **show your work**.
4. For questions marked , provide a well-organized solution in the answer booklet. Use mathematical statements and words to explain all of the steps of your solution. Work out some details in rough on a separate piece of paper before writing your finished solution.
5. Diagrams are *not* drawn to scale. They are intended as aids only.
6. While calculators may be used for numerical calculations, other mathematical steps must be shown and justified in your written solutions, and specific marks may be allocated for these steps. For example, while your calculator might be able to find the x-intercepts of the graph of an equation like $y = x^3 - x$, you should show the algebraic steps that you used to find these numbers, rather than simply writing these numbers down.

A Note about Bubbling

Please make sure that you have correctly coded your name, date of birth and grade on the Student Information Form, and that you have answered the question about eligibility.

1. (a) What is the value of a for which $(a-1)+(2a-3)=14$?

   (b) What are the two values of c for which $(c^2-c)+(2c-3)=9$?

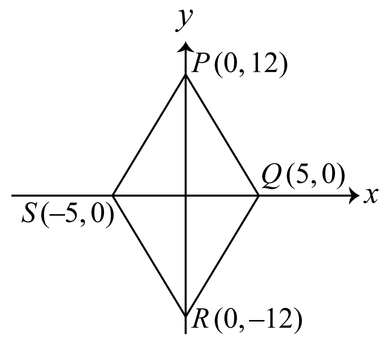
   (c) Determine all values of x for which $\dfrac{1}{x^2}+\dfrac{3}{2x^2}=10$.

2. (a) What is the sum of the digits of the integer equal to $(10^3+1)^2$?

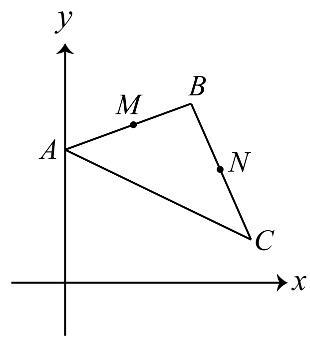
   (b) A bakery sells small and large cookies. Before a price increase, the price of each small cookie is \$1.50 and the price of each large cookie is \$2.00. The price of each small cookie is increased by 10% and the price of each large cookie is increased by 5%. What is the percentage increase in the total cost of a purchase of 2 small cookies and 1 large cookie?

   (c) Qing is twice as old as Rayna. Qing is 4 years younger than Paolo. The average age of Paolo, Qing and Rayna is 13. Determine their ages.

3. (a) In the diagram, $PQRS$ is a quadrilateral. What is its perimeter?



(b) In the diagram, A has coordinates $(0, 8)$. Also, the midpoint of AB is $M(3, 9)$ and the midpoint of BC is $N(7, 6)$. What is the slope of AC?



(c) The parabola with equation $y = -2x^2 + 4x + c$ has vertex $V(1, 18)$. The parabola intersects the y-axis at D and the x-axis at E and F. Determine the area of $\triangle DEF$.

4. (a) If $3(8^x) + 5(8^x) = 2^{61}$, what is the value of the real number x?

(b) For some real numbers m and n, the list $3n^2$, m^2, $2(n + 1)^2$ consists of three consecutive integers written in increasing order. Determine all possible values of m.

5. (a) Chinara starts with the point $(3, 5)$, and applies the following three-step process, which we call $\mathcal{P}$:

Step 1: Reflect the point in the x-axis.
Step 2: Translate the resulting point 2 units upwards.
Step 3: Reflect the resulting point in the y-axis.

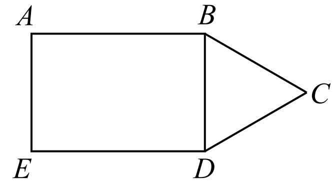
As she does this, the point $(3, 5)$ moves to $(3, -5)$, then to $(3, -3)$, and then to $(-3, -3)$.

Chinara then starts with a different point S_0. She applies the three-step process $\mathcal{P}$ to the point S_0 and obtains the point S_1. She then applies $\mathcal{P}$ to S_1 to obtain the point S_2. She applies $\mathcal{P}$ four more times, each time using the previous output of $\mathcal{P}$ to be the new input, and eventually obtains the point $S_6(-7, -1)$. What are the coordinates of the point S_0?

(b) In the diagram, $ABDE$ is a rectangle, $\triangle BCD$ is equilateral, and AD is parallel to BC. Also, $AE = 2x$ for some real number x.

(i) Determine the length of AB in terms of x.

(ii) Determine positive integers r and s for which $\dfrac{AC}{AD} = \sqrt{\dfrac{r}{s}}$.



6. (a) Suppose that $n > 5$ and that the numbers $t_1, t_2, t_3, \ldots, t_{n-2}, t_{n-1}, t_n$ form an arithmetic sequence with n terms. If $t_3 = 5$, $t_{n-2} = 95$, and the sum of all n terms is 1000, what is the value of n?

(An *arithmetic sequence* is a sequence in which each term after the first is obtained from the previous term by adding a constant, called the common difference. For example, $3, 5, 7, 9$ are the first four terms of an arithmetic sequence.)

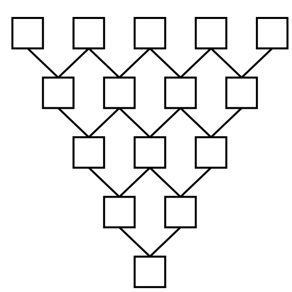
(b) Suppose that a and r are real numbers. A geometric sequence with first term a and common ratio r has 4 terms. The sum of this geometric sequence is $6 + 6\sqrt{2}$. A second geometric sequence has the same first term a and the same common ratio r, but has 8 terms. The sum of this second geometric sequence is $30 + 30\sqrt{2}$. Determine all possible values for a.

(A *geometric sequence* is a sequence in which each term after the first is obtained from the previous term by multiplying it by a non-zero constant, called the common ratio. For example, $3, -6, 12, -24$ are the first four terms of a geometric sequence.)

7. (a) A bag contains 3 green balls, 4 red balls, and no other balls. Victor removes balls randomly from the bag, one at a time, and places them on a table. Each ball in the bag is equally likely to be chosen each time that he removes a ball. He stops removing balls when there are two balls of the same colour on the table. What is the probability that, when he stops, there is at least 1 red ball and at least 1 green ball on the table?

(b) Suppose that $f(a) = 2a^2 - 3a + 1$ for all real numbers a and $g(b) = \log_{\frac{1}{2}} b$ for all $b > 0$. Determine all θ with $0 \le \theta \le 2\pi$ for which $f(g(\sin\theta)) = 0$.

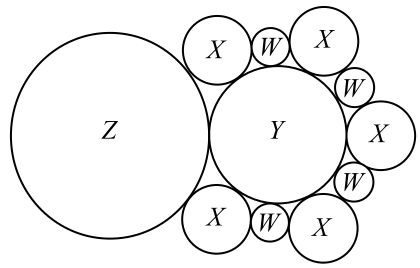
8. (a) Five distinct integers are to be chosen from the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$ and placed in some order in the top row of boxes in the diagram. Each box that is not in the top row then contains the product of the integers in the two boxes connected to it in the row directly above. Determine the number of ways in which the integers can be chosen and placed in the top row so that the integer in the bottom box is 9 953 280 000.



(b) Prove that the integer $\dfrac{(1!)(2!)(3!)\cdots(398!)(399!)(400!)}{200!}$ is a perfect square. (In this fraction, the numerator is the product of the factorials of the integers from 1 to 400, inclusive.)

9. (a) Suppose that $a = 5$ and $b = 4$. Determine all pairs of integers (K, L) for which $K^2 + 3L^2 = a^2 + b^2 - ab$.
(b) Prove that, for all integers K and L, there is at least one pair of integers (a, b) for which $K^2 + 3L^2 = a^2 + b^2 - ab$.
(c) Prove that, for all integers a and b, there is at least one pair of integers (K, L) for which $K^2 + 3L^2 = a^2 + b^2 - ab$.

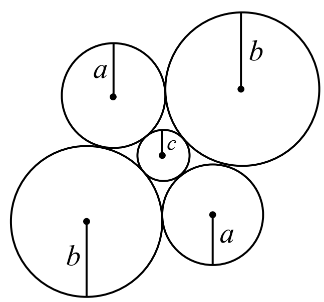
10. (a) In the diagram, eleven circles of four different sizes are drawn. Each circle labelled W has radius 1, each circle labelled X has radius 2, the circle labelled Y has radius 4, and the circle labelled Z has radius r. Each of the circles labelled W or X is tangent to three other circles. The circle labelled Y is tangent to all ten of the other circles. The circle labelled Z is tangent to three other circles. Determine positive integers s and t for which $r = \dfrac{s}{t}$.



(b) Suppose that c is a positive integer. Define $f(c)$ to be the number of pairs (a, b) of positive integers with $c < a < b$ for which two circles of radius a, two circles of radius b, and one circle of radius c can be drawn so that

- each circle of radius a is tangent to both circles of radius b and to the circle of radius c, and
- each circle of radius b is tangent to both circles of radius a and to the circle of radius c,

as shown. Determine all positive integers c for which $f(c)$ is even.



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For students...

Thank you for writing the 2021 Euclid Contest! Each year, more than 260 000 students from more than 80 countries register to write the CEMC's Contests.

If you are graduating from secondary school, good luck in your future endeavours! If you will be returning to secondary school next year, encourage your teacher to register you for the 2021 Canadian Senior Mathematics Contest, which will be written in November 2021.

Visit our website `cemc.uwaterloo.ca` to find

- Free copies of past contests
- Math Circles videos and handouts that will help you learn more mathematics and prepare for future contests
- Information about careers in and applications of mathematics and computer science

For teachers...

Visit our website `cemc.uwaterloo.ca` to

- Obtain information about our 2021/2022 contests
- Look at our free online courseware for high school students
- Learn about our face-to-face workshops and our web resources
- Subscribe to our free Problem of the Week
- Investigate our online Master of Mathematics for Teachers
- Find your school's contest results